

RING COMPLETION OF RIG CATEGORIES

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ABSTRACT. In this paper we offer a solution to the long-standing problem of group completing within the context of rig categories (also known as bimonoidal categories). More precisely, given a rig category \mathcal{R} we construct a natural additive group completion $\bar{\mathcal{R}}$ of \mathcal{R} that retains the multiplicative structure, meaning that it remains a rig category. In other words, it has become a ring category. If we start with a commutative rig category \mathcal{R} (also known as a symmetric bimonoidal category), the additive group completion $\bar{\mathcal{R}}$ can be taken to be a commutative rig category. In an accompanying paper we show how this can be used to prove the conjecture from [BDR] that $K(ku)$ is the K -theory of the rig category of complex vector spaces.

1. INTRODUCTION AND MAIN RESULT

It is well known that multiplicative structure in algebraic K -theory is a delicate matter. In 1980 Thomason demonstrated in [Th2] that, after group completion, the most obvious approaches to multiplicative structures do not make sense. For instance, in the Grayson–Quillen model [G1] for algebraic K -theory of a symmetric monoidal category \mathcal{M} , an object is a pair (A, B) of objects of \mathcal{M} , thought of as “ $A - B$ ”. Then the naïve guess for how to multiply elements is dictated by the rule that $(A - B)(C - D) = (AC + BD) - (AD + BC)$. This, however, does not lead to a decent multiplicative structure: the resulting product is in most situations not functorial.

Several ways around this problem have been developed, but they all have in common that one needs to pass to spectra. The original problem has remained unanswered: can one group complete and keep the multiplicative structure, within the context of symmetric monoidal categories?

We answer this question affirmatively. Our motivation comes from the fact that the positive outcome enters as a step in our proof, [BDRR2], that 2-vector bundles give rise to a geometric cohomology theory of the same sort as elliptic cohomology, or more precisely, to the algebraic K -theory of connective topological K -theory, which by work of Ausoni and the fourth author [A], [AR] is a spectrum of telescopic complexity 2. For this application the alternatives usually provided in spectra were insufficient.

By a rig category (also known as a bimonoidal category), we understand a category with two operations \oplus and \otimes satisfying the axioms of a rig (ring without negative elements) up to coherent natural isomorphisms, see Definitions 2.1 and 2.4 below for details.

Since our permutative structures are to be thought of as additive, we use expressions like $(-\mathcal{M})\mathcal{M}$ where Grayson–Quillen would have written $\mathcal{M}^{-1}\mathcal{M}$.

Theorem 1.1. *Let $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, \otimes, 1_{\mathcal{R}})$ be a small topological rig category. Then there are natural maps of topological rig categories*

$$\mathcal{R} \longleftarrow Z\mathcal{R} \longrightarrow \bar{\mathcal{R}}$$

such that

- (1) $\mathcal{R} \leftarrow Z\mathcal{R}$ induces a weak equivalence of spaces,
- (2) $\bar{\mathcal{R}}$ is a ring category ($\pi_0\bar{\mathcal{R}}$ is a ring),
- (3) the map $\mathrm{Spt} Z\mathcal{R} \rightarrow \mathrm{Spt} \bar{\mathcal{R}}$ is a stable equivalence of spectra, where Spt is any one of the many equivalent infinite loop space machines.
- (4) Assume furthermore that \mathcal{R} satisfies the following conditions:
 - (a) \mathcal{R} is a groupoid (all morphisms in \mathcal{R} are isomorphisms),
 - (b) for every object $X \in \mathcal{R}$ the translation functor $X \oplus (-)$ is faithful.

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Then there is a natural chain of $Z\mathcal{R}$ -module maps, inducing weak equivalences of spaces, connecting $\bar{\mathcal{R}}$ and the Grayson–Quillen model $(-\mathcal{R})\mathcal{R}$ for the group completion of \mathcal{R} .

If \mathcal{R} is a commutative rig category (also known as a symmetric bimonoidal category), so that \oplus and \otimes satisfy the axioms of a commutative rig up to coherent natural isomorphisms, then we can also arrange that the ring completion $\bar{\mathcal{R}}$ is commutative.

Addendum 1.2. *Let \mathcal{R} be a commutative rig category satisfying the conditions of the theorem. Then there are natural maps*

$$\mathcal{R} \longleftarrow Z\mathcal{R} \longrightarrow \bar{\mathcal{R}}$$

of commutative rig categories, so that $\bar{\mathcal{R}}$ is a commutative ring category, and the four statements of the theorem hold.

Important examples:

- If R is a rig and \mathcal{R} is the discrete category (having only identity morphisms) with objects the elements of R , then \mathcal{R} is a rig category. When R is commutative, so is \mathcal{R} .
- The sphere spectrum S is the algebraic K -theory spectrum of the small commutative rig category of finite sets, \mathcal{E} . The objects of \mathcal{E} are the finite sets $\mathbf{n} = \{1, \dots, n\}$ with $n \geq 0$. Here the convention is that $\mathbf{0}$ is the empty set. Morphisms from \mathbf{n} to \mathbf{m} are only non-trivial for $n = m$, and in this case they consist of the symmetric group Σ_n on n letters.
- For a commutative ring A we consider the following small rig category of finitely generated free A -modules, $\mathcal{F}(A)$. Objects of $\mathcal{F}(A)$ are the finitely generated free A -modules A^n for $n \geq 0$. The set of morphisms from A^n to A^m is empty unless $n = m$, and the morphisms from A^n to itself are the A -module automorphisms of A^n , i.e., $GL_n(A)$.
- The case that started this project is the 2-category of 2-vector spaces of Kapranov and Voevodsky [KV], viewed as modules over the commutative rig category \mathcal{V} of complex (Hermitian) vector spaces. Here \mathcal{V} has objects \mathbb{C}^n for $n \geq 0$, and the automorphism space of \mathbb{C}^n is the unitary group $U(n)$.

1.1. Outline of proof. The problem should be approached with some trepidation, since the reasons for the failure of the obvious attempts at a solution to this long-standing problem in K -theory are fairly well hidden. The standard approaches to additive group completion yield models that are symmetric monoidal categories with respect to an additive structure, but which have no meaningful multiplicative structure [Th2]. This failure comes about essentially because commutativity for addition only holds up to isomorphism, and we therefore need to make a model that provides enough room to circumvent this problem. Our solution comes in the form of a graded construction, $G\mathcal{R}$, closely related to iterations of the Grayson–Quillen model. It is a J -shaped diagram of symmetric monoidal categories, where the indexing category $J = I \int Q$ is a certain permutative category over the category I of finite sets $\mathbf{n} = \{1, \dots, n\}$ and injections. The diagram $G\mathcal{R}$ has cubes of products of \mathcal{R} ’s at each node, thought of as “alternating sums” in \mathcal{R} with summands spread out over the corners of the cube, and with sign determined by position. See Section 3 for examples and pictures in low dimensions. The edges in the diagram are either diagonals or injections into faces of cubes. The graded multiplication

$$G\mathcal{R}(j_1) \times G\mathcal{R}(j_2) \rightarrow G\mathcal{R}(j_1 + j_2)$$

is defined by multiplying two cubes together to get a bigger cube, with all possible multiplications of the original cubes spread out over the bigger cube. For instance, the product of the two one-cubes (A, B) and (C, D) is something like the two-cube $\begin{pmatrix} AC & AD \\ BC & BD \end{pmatrix}$.

This multiplication is well-defined. The problem one usually encounters does not appear, essentially because we have spread all the products out over the cubes, and not added together the “positive” and “negative” entries in some order or other.

From a homotopy theoretic point of view, the crucial information lies in the fact that for each $\mathbf{n} \in I$, the homotopy colimit of the associated spectra of the $G\mathcal{R}$ -diagram over \mathbf{n} is equivalent to the spectrum associated with \mathcal{R} . For instance, if $n = 1$, the diagram over $\mathbf{1}$ is $0 \leftarrow \mathcal{R} \rightarrow \mathcal{R} \times \mathcal{R}$ where the second map is the diagonal. Hence, the homotopy colimit of associated spectra is the “mapping cone of the diagonal” and so again is a model for the spectrum associated with \mathcal{R} . From a categorical point of view, the possibility to flip the factors in $\mathcal{R} \times \mathcal{R}$ gives that the passage to spectra is unnecessary, since this flip induces desired “negative path components”, without having to stabilize.

We use Thomason’s homotopy colimit in symmetric monoidal categories [Th3] to transform graded rig categories into rig categories (see Proposition 3.2 and Lemma 5.2). Zeros are troublesome (few

symmetric monoidal categories are “well pointed”), and must be handled with care. This gives rise to the intermediate rig $Z\mathcal{R}$ mentioned in Theorem 1.1.

1.2. Plan. The structure of the paper is as follows. We discuss graded versions of bipermutative and strictly bimonoidal categories and their morphisms in Section 2. In Section 3 we identify the construction $G\mathcal{R}$ mentioned above as a bipermutative (resp. strictly bimonoidal) category that is graded over the category J .

Thomason’s homotopy colimit of symmetric monoidal categories is defined in a non-unital (or zeroless) setting. We extend this to the unital setting by constructing a derived version of it in Section 4, and in Section 5 we show that the homotopy colimit of a graded bipermutative (resp. strictly bimonoidal) category is almost a bipermutative (resp. strictly bimonoidal) category — it only lacks a zero. Section 6 describes how the results obtained so far combine to lead to a multiplicative group completion of (symmetric) bimonoidal categories. This ring completion is given in Theorem 6.4.

Most of this paper appeared earlier as the first part of a preprint with the title “Two-vector bundles define a form of elliptic cohomology”. Some of our readers felt that that title hid the result on rig categories explained in the current paper, which only appeared as a part of the proof of the main theorem. Hence we now offer the multiplicative group completion result separately, and ask those readers also interested in our main application to turn to [BDRR2] for a fuller account of the background. One should note that there was a mathematical error in the preprint: the map T in the purported proof of Lemma 3.7(2) is not well defined, and so the version of the iterated Grayson–Quillen model used there might not have the right homotopy type.

A piece of notation: if \mathcal{C} is any small category, then the expression $X \in \mathcal{C}$ is short for “ X is an object of \mathcal{C} ” and likewise for morphisms and diagrams. Displayed diagrams commute unless the contrary is stated explicitly.

2. BIPERMUTATIVE AND RIG CATEGORIES

For the definition of a permutative category see for instance [EM, 3.1] or [M1, §4]; compare also [ML, XI.1]. Since our permutative categories are typically going to be the underlying additive symmetric monoidal categories of categories with some further structure, we call the neutral element “zero” or simply 0.

We consider lax and strict symmetric monoidal functors F between two permutative categories $(\mathcal{M}, \oplus, 0_{\mathcal{M}}, \tau_{\mathcal{M}})$ and $(\mathcal{N}, \oplus, 0_{\mathcal{N}}, \tau_{\mathcal{N}})$. A *lax symmetric monoidal functor* is a functor F in the sense of [ML, XI.2], *i.e.*, there are morphisms

$$f_{a,a'}: F(a) \oplus F(a') \rightarrow F(a \oplus a')$$

for all objects $a, a' \in \mathcal{M}$ that are natural in a and a' , there is a morphism

$$n: 0_{\mathcal{N}} \rightarrow F(0_{\mathcal{M}}),$$

and these structure maps fulfill the coherence conditions that are spelled out in [ML, XI.2]; in particular

$$\begin{array}{ccc} F(a) \oplus F(a') & \xrightarrow{f_{a,a'}} & F(a \oplus a') \\ \tau_{\mathcal{N}}(F(a), F(a')) \downarrow & & \downarrow F(\tau_{\mathcal{M}}(a, a')) \\ F(a') \oplus F(a) & \xrightarrow{f_{a',a}} & F(a' \oplus a) \end{array}$$

commutes for all $a, a' \in \mathcal{M}$. Let Perm be the category of small permutative categories and lax symmetric monoidal functors.

We might say that f is a binatural transformation, *i.e.*, a natural transformation of functors $\mathcal{M} \times \mathcal{M} \rightarrow \mathcal{N}$. Here “bi-” refers to the two variables, and should not be confused with the “bi-” in bipermutative, which refers to the two operations \oplus and \otimes .

A *strict symmetric monoidal functor* has furthermore to satisfy that the morphisms $f_{a,a'}$ and n are identities, so that

$$F(a \oplus a') = F(a) \oplus F(a') \quad \text{and} \quad F(0_{\mathcal{M}}) = 0_{\mathcal{N}}$$

[ML, XI.2]. We denote the category of small permutative categories and strict symmetric monoidal functors by Strict .

Since any symmetric monoidal category is naturally equivalent to a permutative category, we lose no generality by only considering permutative categories. Note that we consider the unital situation, unless explicitly stated otherwise.

Roughly speaking, a rig category \mathcal{R} consists of a symmetric monoidal category $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, \tau_{\mathcal{R}})$ together with a functor $\mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ called “multiplication” and denoted by \otimes or \cdot . Note that the multiplication is not a map of monoidal categories. The multiplication has a unit $1_{\mathcal{R}} \in \mathcal{R}$, multiplying by $0_{\mathcal{R}}$ is the zero map, multiplying with $1_{\mathcal{R}}$ is the identity map, and the multiplication is (left and right) distributive over \oplus up to appropriate coherencies. If we pose the additional requirement that our rig categories are commutative (up to coherent isomorphisms), then this coincides with what is often called a symmetric bimonoidal category. Laplaza spelled out the coherence conditions in [L, pp. 31–35].

According to [M2, VI, Proposition 3.5] any commutative rig category is equivalent in the appropriate sense to a “bipermutative category”, and a similar rigidification result holds for bimonoidal categories. Our main theorem and its addendum are equivalent to the statements with “rig category” and “commutative rig category” replaced by “strictly bimonoidal category” and “bipermutative category”, respectively, and it is the latter statements that we prove. The reader will find the axioms for a bipermutative category in Definition 2.1 below as the special case of a “0-graded bipermutative category”, where 0 is the one-point category.

Otherwise one may for instance consult [EM, 3.6]. Note that we demand strict right distributivity. One word of warning: in [EM], Elmendorf and Mandell’s left distributivity law is precisely what we (and [M2, VI, Definition 3.3]) call the right distributivity law.

We will focus on the bipermutative case in the course of this paper and indicate what has to be adjusted in the strictly bimonoidal case.

If \mathcal{R} is a small rig category such that $\pi_0(\mathcal{R})$ is a ring (has additive inverses), then we call \mathcal{R} a *ring category*. Elmendorf and Mandell’s ring categories are not ring categories in our sense, but non-commutative rig categories. In the course of this paper we have to resolve rig categories simplicially. If \mathcal{R} is a small simplicial rig category such that $\pi_0(|\mathcal{R}|)$ is a ring, then we call \mathcal{R} a *simplicial ring category* (even though it is usually not a simplicial object in the category of ring categories).

If \mathcal{R} is a strictly bimonoidal category, a left \mathcal{R} -module is a permutative category \mathcal{M} together with a multiplication $\mathcal{R} \times \mathcal{M} \rightarrow \mathcal{M}$ that is strictly associative and coherently distributive as spelled out in [EM, 9.1.1].

2.1. J -graded bipermutative categories and strictly bimonoidal categories. We want to additively group complete a rig category \mathcal{R} , in such a way that the outcome still possesses a multiplicative structure, *i.e.*, so as to produce a ring category. There are constructions for additive group completions of \mathcal{R} , *e.g.* the Grayson–Quillen construction $(-\mathcal{R})\mathcal{R}$, but they are known to have bad multiplicative behavior [Th2].

The following definition of a J -graded bipermutative category is designed to axiomatize the key properties of the functor $G\mathcal{R}$ described in Section 3, and simultaneously to generalize the definition of a bipermutative category (as the case $J = 0$). More generally, we could have introduced graded rig categories, generalizing rig categories, but this would have led to an even more cumbersome definition. We will therefore always assume that the input \mathcal{R} to our machinery has been transformed to an equivalent bipermutative or strictly bimonoidal category before we start.

Definition 2.1. Let $(J, +, 0, c_J)$ be a small permutative category. A J -graded bipermutative category is a functor X from J to the category **Strict** of small permutative categories and strict symmetric monoidal functors, together with the following data and subject to the following conditions, where we denote the permutative structure of $X(i)$ by $(X(i), \oplus, 0_i, \gamma_{\oplus})$.

(1) A functor

$$\otimes: X(i) \times X(j) \rightarrow X(i + j),$$

i.e., for all $(A, B) \in X(i) \times X(j)$ there is an object $A \otimes B$ in $X(i + j)$ and for any pair of morphisms, $f: A \rightarrow A'$ and $g: B \rightarrow B'$, there is a morphism $f \otimes g: A \otimes B \rightarrow A' \otimes B'$ satisfying the usual requirements.

We require that for any pair of morphisms in J , $\varphi: i \rightarrow k$ and $\psi: j \rightarrow \ell$, the following diagram commutes:

$$\begin{array}{ccc} X(i) \times X(j) & \xrightarrow{\otimes} & X(i + j) \\ X(\varphi) \times X(\psi) \downarrow & & \downarrow X(\varphi + \psi) \\ X(k) \times X(\ell) & \xrightarrow{\otimes} & X(k + \ell). \end{array}$$

- (2) An object 1 of $X(0)$ such that the composition of the inclusion $\{1\} \times X(j) \rightarrow X(0) \times X(j)$ followed by $\otimes: X(0) \times X(j) \rightarrow X(0+j) = X(j)$ equals the projection isomorphism $\{1\} \times X(j) \cong X(j)$, and likewise for the map from $X(j) \times \{1\}$.
- (3) Isomorphisms

$$\gamma_{\otimes} = \gamma_{\otimes}^{A,B}: A \otimes B \longrightarrow X(c_J^{j,i})(B \otimes A)$$

in $X(i+j)$, for all $A \in X(i)$ and $B \in X(j)$, such that

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\gamma_{\otimes}^{A,B}} & X(c_J^{j,i})(B \otimes A) \\ f \otimes g \downarrow & & \downarrow X(c_J^{j,i})(g \otimes f) \\ A' \otimes B' & \xrightarrow{\gamma_{\otimes}^{A',B'}} & X(c_J^{j,i})(B' \otimes A') \end{array}$$

commutes and $X(c_J^{j,i})(\gamma_{\otimes}^{B,A}) \circ \gamma_{\otimes}^{A,B}$ is equal to the identity on $A \otimes B$:

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\text{id}_{A \otimes B}} & X(c_J^{j,i})X(c_J^{i,j})(A \otimes B) \\ & \searrow \gamma_{\otimes}^{A,B} & \nearrow X(c_J^{j,i})(\gamma_{\otimes}^{B,A}) \\ & X(c_J^{j,i})(B \otimes A) & \end{array}$$

For φ and ψ as above we require that

$$X(\varphi + \psi)(\gamma_{\otimes}^{A,B}) = \gamma_{\otimes}^{X(\varphi)(A), X(\psi)(B)}.$$

In other words, the multiplicative twist γ_{\otimes} is natural in i and j .

In addition, $\gamma_{\otimes}^{A,1}$ and $\gamma_{\otimes}^{1,A}$ agree with the identity morphism on A for all objects A .

- (4) The composition \otimes is associative and the diagram

$$\begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{\gamma_{\otimes}^{A \otimes B, C}} & X(c_J^{k,i+j})(C \otimes A \otimes B) \\ \text{id} \otimes \gamma_{\otimes}^{B,C} \downarrow & & \downarrow X(c_J^{k,i+j})(\gamma_{\otimes}^{C,A \otimes B}) \\ X(\text{id} + c_J^{k,j})(A \otimes C \otimes B) & \equiv & X(c_J^{k,i+j})X(c_J^{i,k} + \text{id})(A \otimes C \otimes B) \end{array}$$

commutes for all objects (compare [ML, p. 254, (7a)]).

- (5) For each $i \in J$ the zero object 0_i annihilates everything multiplicatively, *i.e.*, $\{0_i\} \times X(j) \rightarrow X(i) \times X(j) \rightarrow X(i+j)$ is the constant map to 0_{i+j} . Here the first map is the inclusion and the second is \otimes .
- (6) Right distributivity holds strictly, *i.e.*,

$$\begin{array}{ccc} (X(i) \times X(i)) \times X(j) & \xrightarrow{\oplus \times \text{id}} & X(i) \times X(j) \\ \downarrow \Delta & & \downarrow \otimes \\ (X(i) \times X(j)) \times (X(i) \times X(j)) & & \\ \downarrow \otimes \times \otimes & & \downarrow \oplus \\ X(i+j) \times X(i+j) & \xrightarrow{\oplus} & X(i+j) \end{array}$$

commutes, where \oplus is the monoidal structure and Δ is the diagonal on $X(j)$ combined with the identity on $X(i) \times X(i)$ followed by a twist. We denote these instances of identities by d_r .

- (7) The left distributivity transformation, d_ℓ , is given in terms of d_r and γ_{\otimes} as

$$d_\ell = \gamma_{\otimes} \circ d_r \circ (\gamma_{\otimes} \oplus \gamma_{\otimes}).$$

(Here, we suppress the twist $X(c_J)$ in the notation.) Thus, for all i and j and $A \in X(i)$, $B, C \in X(j)$ the following diagram defines d_ℓ :

$$\begin{array}{ccc}
A \otimes B \oplus A \otimes C & \xrightarrow{\gamma_\otimes^{A,B} \oplus \gamma_\otimes^{A,C}} & X(c_J^{j,i})(B \otimes A) \oplus X(c_J^{j,i})(C \otimes A) \\
\downarrow d_\ell & & \parallel \\
A \otimes (B \oplus C) & & X(c_J^{j,i})(B \otimes A \oplus C \otimes A) \\
\parallel & & \downarrow X(c_J^{j,i})(d_r) = X(c_J^{j,i})(\text{id}) = \text{id} \\
X(c_J^{j,i})X(c_J^{i,j})(A \otimes (B \oplus C)) & \xleftarrow{X(c_J^{j,i})(\gamma_\otimes^{B \oplus C, A})} & X(c_J^{j,i})((B \oplus C) \otimes A).
\end{array}$$

(8) The diagram

$$\begin{array}{ccc}
(A \otimes B) \oplus (A \otimes B') & \xrightarrow{d_\ell} & A \otimes (B \oplus B') \\
\downarrow \gamma_\oplus & & \downarrow \text{id} \otimes \gamma_\oplus \\
(A \otimes B') \oplus (A \otimes B) & \xrightarrow{d_\ell} & A \otimes (B' \oplus B)
\end{array}$$

commutes for all objects. The analogous diagram for d_r also commutes.

Due to the definition of d_ℓ in terms of γ_\otimes and the identity d_r , it suffices to demand that $\gamma_\oplus \circ (\gamma_\otimes \oplus \gamma_\otimes) = (\gamma_\otimes \oplus \gamma_\otimes) \circ \gamma_\oplus$ and $(\gamma_\oplus \otimes \text{id}) \circ \gamma_\otimes = \gamma_\otimes \circ (\text{id} \otimes \gamma_\oplus)$.

(9) The distributivity transformations are associative, *i.e.*, the diagram

$$\begin{array}{ccc}
(A \otimes B \otimes C) \oplus (A \otimes B \otimes C') & & \\
\downarrow d_\ell & \searrow d_\ell & \\
A \otimes ((B \otimes C) \oplus (B \otimes C')) & \xrightarrow{\text{id} \otimes d_\ell} & A \otimes B \otimes (C \oplus C')
\end{array}$$

commutes for all objects.

(10) The following pentagon diagram commutes

$$\begin{array}{ccccc}
& & (A \otimes (B \oplus B')) \oplus (A' \otimes (B \oplus B')) & & \\
& \nearrow d_\ell \oplus d_\ell & & \searrow d_r & \\
(A \otimes B) \oplus (A \otimes B') \oplus (A' \otimes B) \oplus (A' \otimes B') & & & & (A \oplus A') \otimes (B \oplus B') \\
\downarrow \text{id} \oplus \gamma_\oplus \oplus \text{id} & & & & \uparrow d_\ell \\
(A \otimes B) \oplus (A' \otimes B) \oplus (A \otimes B') \oplus (A' \otimes B') & & & & \\
& \searrow d_r \oplus d_r & & \nearrow & \\
& ((A \oplus A') \otimes B) \oplus ((A \oplus A') \otimes B') & & &
\end{array}$$

for all objects $A, A' \in X(i)$ and $B, B' \in X(j)$.

Remark 2.2. Notice that in Definition 2.1, the condition (1) only says that we have a natural transformation

$$\otimes: X \times X \Rightarrow X \circ +$$

of functors $J \times J \rightarrow \text{Cat}$, and condition (3) demands a modification

$$\begin{array}{ccc}
X \times X & \xrightarrow{\text{Cat}} & (X \times X) \circ tw_J \\
\downarrow \otimes & \gamma_\otimes \Rightarrow & \downarrow \otimes \\
X \circ + & \xleftarrow{X(c_J)} & X \circ + \circ tw_J
\end{array}$$

where c_{Cat} is the symmetric structure on Cat (with respect to product) and tw_J is the interchange of factors on $J \times J$.

In the following we will denote a J -graded bipermutative category $X: J \rightarrow \text{Strict}$ by X^\bullet if the category J is clear from the context. For the one-point category $J = 0$, a J -graded bipermutative category is the same as a bipermutative category. Thus every J -graded bipermutative category X^\bullet comes with a bipermutative category $X(0)$, and X^\bullet can be viewed as a functor $J \rightarrow X(0)$ -modules.

Example 2.3. We consider the small bipermutative category of finite sets, whose objects are the finite sets of the form $\mathbf{n} = \{1, \dots, n\}$ for $n \geq 0$ and $\mathbf{0} = \emptyset$, and whose morphisms are functions.

Disjoint union of sets gives rise to a permutative structure

$$\mathbf{n} \oplus \mathbf{m} := \mathbf{n} \sqcup \mathbf{m}$$

and we identify $\mathbf{n} \sqcup \mathbf{m}$ with $\mathbf{n} + \mathbf{m}$. For functions $f: \mathbf{n} \rightarrow \mathbf{n}'$ and $g: \mathbf{m} \rightarrow \mathbf{m}'$ we define $f \oplus g$ as the map on the disjoint union $f \sqcup g$ which we will denote by $f + g$. The additive twist c_\oplus is given by the shuffle maps

$$\chi(n, m): \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{m} + \mathbf{n}$$

with

$$\chi(n, m)(i) = \begin{cases} m + i & \text{for } i \leq n \\ i - n & \text{for } i > n. \end{cases}$$

Multiplication of sets is defined via

$$\mathbf{n} \otimes \mathbf{m} := \mathbf{nm}.$$

If we identify the element $(i - 1) \cdot m + j$ in \mathbf{nm} with the pair (i, j) with $i \in \mathbf{n}$ and $j \in \mathbf{m}$, then the function $f \otimes g$ is given as

$$(i, j) \mapsto (f(i), g(j)),$$

and the multiplicative twist

$$c_\otimes: \mathbf{n} \otimes \mathbf{m} \longrightarrow \mathbf{m} \otimes \mathbf{n}$$

sends (i, j) to (j, i) . The empty set is a strict unit for the addition and the set $\mathbf{1}$ is a strict unit for the multiplication. Right distributivity is the identity and the left distributivity law is given by the resulting permutation

$$d_\ell = c_\otimes \circ d_r \circ (c_\otimes \oplus c_\otimes).$$

For later reference we denote this instance of d_ℓ by ξ .

Considering only the subcategory of bijections, instead of arbitrary functions, results in the bipermutative category of finite sets \mathcal{E} that we discussed in the introduction. Later, we will consider the bipermutative category of finite sets and surjective functions.

Definition 2.4. A J -graded strictly bimonoidal category is a functor $X: J \rightarrow \text{Strict}$ to the category of permutative categories and strict symmetric monoidal functors, satisfying the conditions of Definition 2.1, except that we do not require the existence of the natural isomorphism γ_\otimes , and the left distributivity isomorphism d_ℓ is not given in terms of d_r . Axiom (7) of Definition 2.1 has to be replaced by the following condition.

(7') The diagram

$$\begin{array}{ccc} A \otimes B \otimes C \oplus A \otimes B' \otimes C & \xrightarrow{d_r} & (A \otimes B \oplus A \otimes B') \otimes C \\ d_\ell \downarrow & & \downarrow d_\ell \otimes \text{id} \\ A \otimes (B \otimes C \oplus B' \otimes C) & \xrightarrow{\text{id} \otimes d_r} & A \otimes (B \oplus B') \otimes C \end{array}$$

commutes for all objects.

In the J -graded bipermutative case condition (7') follows from the other axioms.

Definition 2.5. A lax morphism of bipermutative categories, $g: X \rightarrow Y$, is a lax symmetric monoidal functor from $(X, \oplus, 0_X, c_\oplus)$ to $(Y, \oplus, 0_Y, c_\oplus)$ together with a structure of a lax symmetric monoidal functor from $(X, \otimes, 1_X, c_\otimes)$ to $(Y, \otimes, 1_Y, c_\otimes)$, that respects the distributivity laws.

We therefore have a binatural transformation η_\oplus from $(- \oplus -) \circ (g, g)$ to $g \circ (- \oplus -)$, i.e.,

$$\eta_\oplus = \eta_\oplus(A, B): g(A) \oplus g(B) \rightarrow g(A \oplus B) \text{ for } A, B \in X,$$

and a corresponding binatural transformation from $(- \otimes -) \circ (g, g)$ to $g \circ (- \otimes -)$

$$\eta_\otimes = \eta_\otimes(A, B): g(A) \otimes g(B) \rightarrow g(A \otimes B) \text{ for } A, B \in X,$$

and we require that these interact with c_{\oplus} and c_{\otimes} and that the following diagram (and the analogous one for d_{ℓ}) commutes

$$\begin{array}{ccc} g(A) \otimes g(B) \oplus g(A') \otimes g(B) & \xrightarrow{d_r = \text{id}} & (g(A) \oplus g(A')) \otimes g(B) \xrightarrow{\eta_{\oplus} \otimes \text{id}} g(A \oplus A') \otimes g(B) \\ \eta_{\otimes} \oplus \eta_{\otimes} \downarrow & & \downarrow \eta_{\otimes} \\ g(A \otimes B) \oplus g(A' \otimes B) & \xrightarrow{\eta_{\oplus}} & g(A \otimes B \oplus A' \otimes B) \xrightarrow{g(d_r) = \text{id}} g((A \oplus A') \otimes B) \end{array}$$

for all objects $A, A', B \in X$, *i.e.*, we have

$$\eta_{\oplus} \circ (\eta_{\otimes} \oplus \eta_{\otimes}) = \eta_{\otimes} \circ (\eta_{\oplus} \otimes \text{id})$$

and

$$g(\gamma_{\otimes} \circ (\gamma_{\otimes} \oplus \gamma_{\otimes})) \circ \eta_{\oplus} \circ (\eta_{\otimes} \oplus \eta_{\otimes}) = \eta_{\otimes} \circ (\text{id} \otimes \eta_{\oplus}) \circ \gamma_{\otimes} \circ (\gamma_{\otimes} \oplus \gamma_{\otimes}).$$

For a *lax morphism of strictly bimonoidal categories* we demand that g is lax monoidal with respect to \otimes , lax symmetric monoidal with respect to \oplus and that

$$g(d_{\ell}) \circ \eta_{\oplus} \circ (\eta_{\otimes} \oplus \eta_{\otimes}) = \eta_{\otimes} \circ (\text{id} \otimes \eta_{\oplus}) \circ d_{\ell} \quad \text{and} \quad g(d_r) \circ \eta_{\oplus} \circ (\eta_{\otimes} \oplus \eta_{\otimes}) = \eta_{\otimes} \circ (\eta_{\oplus} \otimes \text{id}) \circ d_r.$$

Definition 2.6. A *lax morphism of J -graded bipermutative categories*, $g: X^{\bullet} \rightarrow Y^{\bullet}$, consists of a natural transformation g from X^{\bullet} to Y^{\bullet} that is compatible with the bifunctors \oplus, \otimes and the units. In detail, we require that there are transformations η_{\oplus} from $(-\oplus-) \circ (g \times g)$ to $g \circ (-\oplus-)$

$$\begin{array}{ccc} X(i) \times X(j) & \xrightarrow{\oplus} & X(i+j) \\ g(i) \times g(j) \downarrow & \nearrow \eta_{\oplus}^{i,j} & \downarrow g(i+j) \\ Y(i) \times Y(j) & \xrightarrow{\oplus} & Y(i+j), \end{array}$$

and η_{\otimes} from $(-\otimes-) \circ (g, g)$ to $g \circ (-\otimes-)$. These commute with γ_{\oplus} and γ_{\otimes} and they are binatural with respect to i and j and morphisms in $X(i), X(j)$.

The functor g respects the distributivity constraints in that it fulfills

$$\eta_{\oplus} \circ (\eta_{\otimes} \oplus \eta_{\otimes}) = \eta_{\otimes} \circ (\eta_{\oplus} \otimes \text{id})$$

and

$$g(d_{\ell}) \circ \eta_{\oplus} \circ (\eta_{\otimes} \oplus \eta_{\otimes}) = \eta_{\otimes} \circ (\text{id} \otimes \eta_{\oplus}) \circ d_{\ell}.$$

For a *lax morphism of J -graded strictly bimonoidal categories* there is no requirement on g concerning the multiplicative twist γ_{\otimes} .

3. A GRADED BIPERMUTATIVE CATEGORY

We remodel the Grayson–Quillen model [G1] for the group completion of a permutative category to suit our multiplicative needs. We will choose models where a product of two elements in the Grayson–Quillen model will be a 2-dimensional cube (= a square), and thus the terms in a product are spread out in order to avoid the “phoniness” of the multiplication [Th2].

Let I be the usual skeleton of the category of finite sets and injective functions, *i.e.*, its objects are the finite sets $\mathbf{n} = \{1, \dots, n\}$ for $n \geq 0$ with the convention that $\mathbf{0} = \emptyset$, and its morphisms are the injective functions $\varphi: \mathbf{m} \rightarrow \mathbf{n}$ of finite sets. We define the sum of two objects \mathbf{n} and \mathbf{m} to be $\mathbf{n} + \mathbf{m}$ and use the twist maps $\chi(n, m)$ defined in Example 2.3. Then $(I, +, \mathbf{0}, \chi)$ is a permutative category.

3.1. An indexing category. Let $\mathcal{Q}\mathbf{n}$ be the category of pointed subsets $S \subseteq \{-n, \dots, n\}$ (pointed at 0) such that the absolute value function $S \rightarrow \mathbb{Z}$ is injective. For instance $\mathcal{Q}\mathbf{2}$ is the category

$$\begin{array}{ccccc} \{-1, 0, 2\} & \longleftarrow & \{0, 2\} & \longrightarrow & \{0, 1, 2\} \\ \uparrow & & \uparrow & & \uparrow \\ \{-1, 0\} & \longleftarrow & \{0\} & \longrightarrow & \{0, 1\} \\ \downarrow & & \downarrow & & \downarrow \\ \{-2, -1, 0\} & \longleftarrow & \{-2, 0\} & \longrightarrow & \{-2, 0, 1\}. \end{array}$$

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The subcategory of $\mathcal{Q}\mathbf{n}$ of pointed subsets of $\mathbf{n}_+ = \{0, 1, \dots, n\}$ is denoted $\mathcal{P}\mathbf{n}$. One should notice that it is isomorphic to the category of subsets of \mathbf{n} . Accordingly, if $S \subseteq \mathbf{m}_+$ is a pointed subset and $\varphi: \mathbf{m} \rightarrow \mathbf{n}$ is an injection, we take the liberty to write $\varphi(S)$ for the pointed subset of \mathbf{n}_+ which pedants would record, quite correctly, as $\varphi(S - \{0\}) \sqcup \{0\}$.

If $\varphi: \mathbf{m} \rightarrow \mathbf{n} \in I$ is an injection, then $\mathcal{Q}\varphi: \mathcal{Q}\mathbf{m} \rightarrow \mathcal{Q}\mathbf{n}$ is the functor given as follows. First, let $C\varphi = \mathbf{n} - \varphi(\mathbf{m})$ be the complement of the image of φ . Then extend φ to an odd function $\{-m, \dots, m\} \rightarrow \{-n, \dots, n\}$ which we will also call φ , and set $\mathcal{Q}(\varphi)S = \varphi(S) \sqcup C\varphi$. So, for instance if $\varphi: \mathbf{1} \rightarrow \mathbf{2}$ is the injection with $\varphi(1) = 2$, then $C\varphi = \{1\}$ and $\mathcal{Q}\varphi$ is the functor

$$\begin{array}{ccc} \mathcal{Q}\mathbf{1} & = & \{\{-1, 0\} \longleftarrow \{0\} \longrightarrow \{0, 1\}\} \\ \downarrow & & \downarrow \quad \downarrow \quad \downarrow \\ \mathcal{Q}\mathbf{2} & \supseteq & \{\{-2, 0, 1\} \longleftarrow \{0, 1\} \longrightarrow \{0, 1, 2\}\} \end{array}$$

embedding $\mathcal{Q}\mathbf{1}$ into the right hand column in $\mathcal{Q}\mathbf{2}$. Likewise, the injection with $\varphi(1) = 1$ embeds $\mathcal{Q}\mathbf{1}$ into the upper row of $\mathcal{Q}\mathbf{2}$.

If $\psi: \mathbf{k} \rightarrow \mathbf{m} \in I$ we see that $\mathcal{Q}\varphi \circ \mathcal{Q}\psi = \mathcal{Q}(\varphi\psi)$, and so \mathcal{Q} defines a functor $\mathcal{Q}: I \rightarrow \text{Cat}$.

Restricting to only sets with non-negative entries, we get a subfunctor $\mathcal{P} \subseteq \mathcal{Q}$ which may be easier to grasp: if $\varphi: \mathbf{m} \rightarrow \mathbf{n} \in I$, then $\mathcal{P}\varphi: \mathcal{P}\mathbf{m} \rightarrow \mathcal{P}\mathbf{n}$ is the functor sending $S \subseteq \mathbf{m}_+$ to $\varphi(S) \sqcup C\varphi$ where $C\varphi = \mathbf{n} - \varphi(\mathbf{m})$ is the complement of the image of φ .

Consider the Grothendieck construction $I \int \mathcal{Q}$. This is the category with objects pairs (\mathbf{m}, S) with $\mathbf{m} \in I$ and $S \in \mathcal{Q}\mathbf{m}$ and with a morphism $(\mathbf{m}, S) \rightarrow (\mathbf{n}, T)$ consisting of an injection $\varphi: \mathbf{m} \rightarrow \mathbf{n} \in I$ and an inclusion $\mathcal{Q}(\varphi)S \subseteq T$.

Consider the functor $+: \mathcal{Q}\mathbf{m} \times \mathcal{Q}\mathbf{n} \rightarrow \mathcal{Q}(\mathbf{m} + \mathbf{n})$ defined as follows. The inclusions $in_1: \mathbf{m} \rightarrow \mathbf{m} + \mathbf{n}$ and $in_2: \mathbf{n} \rightarrow \mathbf{m} + \mathbf{n}$ are given by $in_1(i) = i$ and $in_2(j) = m + j$, and extending to odd functions we define $S + T$ to be the pointed sum of images $in_1(S) \vee in_2(T) \subseteq \{-m - n, \dots, m + n\}$. So, if $S = \{-1, 0, 2\} \subseteq \{-3, \dots, 3\}$ and $T = \{-2, 0, 1\} \subseteq \{-2, -1, 0, 1, 2\}$, then $S + T = \{-5, -1, 0, 2, 4\} \subseteq \{-5, \dots, 5\}$.

Let $(\mathbf{m}, S), (\mathbf{n}, T) \in I \int \mathcal{Q}$. Then we define $(\mathbf{m}, S) + (\mathbf{n}, T) = (\mathbf{m} + \mathbf{n}, S + T)$, and likewise on morphisms.

Lemma 3.1. *Addition makes $I \int \mathcal{Q}$ and $I \int \mathcal{P}$ into permutative categories.*

3.2. The cube construction. Let \mathcal{M} be a permutative category (with zero). Define a functor $\mathcal{M}_{\mathbf{n}}: \mathcal{P}\mathbf{n} \rightarrow \text{Strict}$ sending a pointed subset $S \subseteq \mathbf{n}_+$ to $\mathcal{M}^{\mathcal{P}S}$, the permutative category of functions from the **set** $\mathcal{P}S$ of pointed subsets of S to \mathcal{M} (i.e., the product of \mathcal{M} with itself indexed over the pointed subsets of S). If $S \subseteq T \subseteq \mathbf{n}_+$ are pointed inclusions, we get a strict symmetric monoidal functor $\mathcal{M}^{\mathcal{P}S} \rightarrow \mathcal{M}^{\mathcal{P}T}$ by sending the object $a = (a_U)_{U \subseteq S} \in \mathcal{M}^{\mathcal{P}S}$ to $(a_{V \cap S})_{V \subseteq T}$, and likewise with morphisms.

For $\mathbf{n} = \mathbf{0}, \mathbf{1}, \mathbf{2}$, the diagrams $\mathcal{M}_{\mathbf{n}}$ have the following shapes

$$\begin{array}{ccccc} \mathcal{M}, & \mathcal{M} \rightarrow \mathcal{M} \times \mathcal{M}, & \text{and} & \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}^{\times 4} \\ & & & \uparrow \quad \uparrow \\ & & & \mathcal{M} \longrightarrow \mathcal{M} \times \mathcal{M} \end{array}$$

where the maps are the appropriate diagonals.

For $\varphi: \mathbf{m} \rightarrow \mathbf{n}$ we define the natural transformation $\mathcal{M}_{\varphi}: \mathcal{M}_{\mathbf{m}} \Rightarrow \mathcal{M}_{\mathbf{n}} \circ \mathcal{P}\varphi$: for $S \in \mathcal{P}\mathbf{m}$ we let $\mathcal{M}_{\varphi}(S)$ be the composite

$$\mathcal{M}_{\mathbf{m}}(S) = \mathcal{M}^{\mathcal{P}S} \cong \mathcal{M}^{\mathcal{P}(\varphi(S))} \rightarrow \mathcal{M}^{\mathcal{P}(\varphi(S) \sqcup C\varphi)} = \mathcal{M}_{\mathbf{n}}(\mathcal{P}\varphi)(S)$$

where the isomorphism is just the reindexation induced by φ , and the map $\mathcal{M}^{\mathcal{P}\varphi(S)} \rightarrow \mathcal{M}^{\mathcal{P}(\varphi(S) \sqcup C\varphi)}$ is the identity on factors indexed by subsets of $\varphi(S)$ and zero on the factors that are not hit by φ . Explicitly,

$$\mathcal{M}_{\varphi}(S)(f)_V = \begin{cases} f_{\varphi^{-1}(V)} & \text{if } V \subseteq \varphi(S) \\ 0 & \text{otherwise} \end{cases}$$

for any morphism $f: a \rightarrow b \in \mathcal{M}^{\mathcal{P}S}$ and $V \subseteq \varphi(S) \sqcup C\varphi$.

For instance, if $\varphi: \mathbf{1} \rightarrow \mathbf{2}$ is given by $\varphi(1) = 2$, then $\mathcal{M}_{\mathbf{1}}(\{0\}) = \mathcal{M} \rightarrow \mathcal{M} \times \mathcal{M} = \mathcal{M}_{\mathbf{2}}(\{0, 1\})$ and $\mathcal{M}_{\mathbf{1}}(\{0, 1\}) = \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}^{\times 4} \cong \mathcal{M}_{\mathbf{2}}(\{0, 1, 2\})$ are given by appropriate inclusions onto factors in products (**not** diagonals! Note in particular that for both inclusions $\mathbf{1} \rightarrow \mathbf{2}$ the associated maps

$\mathcal{M} \rightarrow \mathcal{M} \times \mathcal{M}$ are the inclusion onto the $\{0\}$ -factor, whereas the two maps $\mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}^{\times 4}$ are inclusion onto either – depending on φ – the $\{0\}$ and $\{0, 1\}$, or the $\{0\}$ and $\{0, 2\}$ factors).

We see that for all $S \subseteq T \subseteq \mathbf{n}$, the diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathcal{P}S} & \longrightarrow & \mathcal{M}^{\mathcal{P}T} \\ \downarrow & & \downarrow \\ \mathcal{M}^{\mathcal{P}(\varphi(S) \sqcup C\varphi)} & \longrightarrow & \mathcal{M}^{\mathcal{P}(\varphi(T) \sqcup C\varphi)} \end{array}$$

commutes, sending $a \in \mathcal{M}^{\mathcal{P}S}$ both ways to $W \mapsto a_{\varphi^{-1}(W) \cap S} = a_{\varphi^{-1}(W \cap T)}$ if $W \subseteq \varphi(T)$, and 0 otherwise.

If $\psi: \mathbf{k} \rightarrow \mathbf{m} \in I$ then we have an equality of natural transformations $\mathcal{M}_{\varphi\psi} = \mathcal{M}_\varphi \mathcal{M}_\psi: \mathcal{M}_{\mathbf{k}} \Rightarrow \mathcal{M}_{\mathbf{n}} \circ \mathcal{P}(\varphi\psi)$ in

$$\begin{array}{ccc} & \mathcal{P}\mathbf{k} & \\ \mathcal{P}(\varphi\psi) \swarrow \mathcal{P}\psi & \downarrow & \searrow \mathcal{M}_{\mathbf{k}} \\ & \mathcal{P}\mathbf{m} & \xrightarrow{\mathcal{M}_{\mathbf{m}}} \text{Strict} : \\ \mathcal{P}\varphi \searrow & \downarrow & \swarrow \mathcal{M}_{\mathbf{n}} \\ & \mathcal{P}\mathbf{n} & \end{array}$$

both are represented by the functors $\mathcal{M}^{\mathcal{P}S} \rightarrow \mathcal{M}^{\mathcal{P}(\varphi\psi(S) \sqcup C(\varphi\psi))}$ sending a to $V \mapsto a_{(\varphi\psi)^{-1}V}$ if $V \subseteq \varphi\psi(S)$ and zero otherwise. Thus \mathcal{M} can be viewed as a natural transformation from the functor \mathcal{P} to the constant functor Strict .

The natural transformation $\mathcal{M}: \mathcal{P} \Rightarrow \text{Strict}$ extends to a natural transformation $\mathcal{M}: \mathcal{Q} \Rightarrow \text{Strict}$ by declaring that $\mathcal{M}_{\mathbf{m}}(S) = 0$ if S contains negative elements.

The first three diagrams now look like

$$\begin{array}{ccccc} & & 0 & \longleftarrow & \mathcal{M} \times \mathcal{M} & \longrightarrow & \mathcal{M}^{\times 4} \\ & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{M}, & 0 \leftarrow \mathcal{M} \rightarrow \mathcal{M} \times \mathcal{M}, & \text{and} & 0 & \longleftarrow & \mathcal{M} & \longrightarrow & \mathcal{M} \times \mathcal{M} \\ & & & \downarrow & & \downarrow & & \downarrow \\ & & & 0 & \longleftarrow & 0 & \longrightarrow & 0 \end{array}$$

Another way of saying that we have a natural transformation $\mathcal{Q} \Rightarrow \text{Strict}$ is to say that we have a functor $I \int \mathcal{M}: I \int \mathcal{Q} \rightarrow I \int \text{Strict} \cong I \times \text{Strict}$.

Projecting to the second factor, $I \int \mathcal{M}$ gives rise to a functor

$$G\mathcal{M}: I \int \mathcal{Q} \rightarrow \text{Strict}.$$

Explicitly, $G\mathcal{M}(\mathbf{m}, S) = \mathcal{M}_{\mathbf{m}}(S)$ (which is $\mathcal{M}^{\mathcal{P}S}$ if S contains no negatives, and zero otherwise) and if $\varphi: \mathbf{m} \rightarrow \mathbf{n} \in I$ and i is an inclusion $\mathcal{Q}(\varphi)S \subseteq T \in \mathcal{Q}\mathbf{n}$, then $G\mathcal{M}(\varphi, i): \mathcal{M}_{\mathbf{m}}(S) \rightarrow \mathcal{M}_{\mathbf{n}}(T)$ is the composite of $G\mathcal{M}(\varphi, 1) = \mathcal{M}_\varphi S: \mathcal{M}_{\mathbf{m}}(S) \rightarrow \mathcal{M}_{\mathbf{n}}(\mathcal{Q}(\varphi)S)$ and $G\mathcal{M}(1, i) = \mathcal{M}_{\mathbf{n}}(i): \mathcal{M}_{\mathbf{n}}(\mathcal{Q}(\varphi)S) \rightarrow \mathcal{M}_{\mathbf{n}}(T)$.

3.3. Multiplicative structure. Since the diagram $G\mathcal{M}: I \int \mathcal{Q} \rightarrow \text{Strict}$ is so simple, only consisting of diagonals and inclusions on factors of products, algebraic structure on \mathcal{M} is easily transferred to $G\mathcal{M}$. In particular

Proposition 3.2. *If \mathcal{R} is a strictly bimonoidal category, then $G\mathcal{R}$ is an $I \int \mathcal{Q}$ -graded strictly bimonoidal category. If \mathcal{R} is a bipermutative category, then $G\mathcal{R}$ is an $I \int \mathcal{Q}$ -graded bipermutative category.*

Proof. Let $a \in G\mathcal{R}(\mathbf{m}, S)$ and $b \in G\mathcal{R}(\mathbf{n}, T)$. If either S or T contain negatives, then $a \otimes b = 0 \in G\mathcal{R}(\mathbf{m} + \mathbf{n}, S + T)$. If neither S nor T contain negatives, then $(a \otimes b)_{U+V} = a_U \otimes b_V$ for $U \subseteq S$ and $V \subseteq T$. Likewise for morphisms. Since everything is defined pointwise, the multiplicative structure on \mathcal{R} forces all the axioms of a graded $I \int \mathcal{Q}$ -graded bipermutative category or strictly bimonoidal category on $G\mathcal{R}$. \square

4. HOCOLIM-LEMMATA

We briefly recall Thomason's homotopy colimit construction in the case of a functor from a small category J to the category Perm^{nz} of permutative categories without zero objects and lax symmetric monoidal functors.

4.1. The non-unital case. Let $X: J \rightarrow \text{Perm}^{\text{nz}}$ be a functor. An object in $\text{hocolim}_J X$ is an expression like $n[(a_1, X_1), \dots, (a_n, X_n)]$ where $n \geq 1$ is a natural number, the a_i are objects of J and the X_i are objects of $X(a_i)$. A morphism from $n[(a_1, X_1), \dots, (a_n, X_n)]$ to $m[(b_1, Y_1), \dots, (b_m, Y_m)]$ consists of three parts: a surjection ψ from the set $\{1, \dots, n\}$ to $\{1, \dots, m\}$, morphisms $\ell_i: a_i \rightarrow b_{\psi(i)}$ for $1 \leq i \leq n$ and morphisms ϱ_j in $X(b_j)$ from $\bigoplus_{\psi(i)=j} X(\ell_i)(X_i)$ to Y_j . By abuse of notation, we will write $(\psi, \ell_i, \varrho_j)$ to signify this morphism.

The category $\text{hocolim}_J X$ is permutative (without a zero) if one defines the addition to be given by concatenation (compare [Th3, p. 1632]).

As a matter of fact, if $\text{Strict}^{\text{nz}}$ is the subcategory of Perm^{nz} with all objects, but with strict monoidal functors as morphisms, the universal property in [Th3, pp. 1632–1633] says that hocolim_J is left adjoint to the composite functor $\text{Strict}^{\text{nz}} \rightarrow \text{Perm}^{\text{nz}} \rightarrow (\text{Perm}^{\text{nz}})^J$, where the first functor is the forgetful one and the second functor assigns the constant J -diagram (= functor from J).

Here $(\text{Perm}^{\text{nz}})^J$ is the category whose objects are functors $J \rightarrow \text{Perm}^{\text{nz}}$, and whose morphisms are left lax natural transformations.

Let Cat denote the category of all small categories. Recall the free functor $P: \text{Cat} \rightarrow \text{Strict}^{\text{nz}}$ with $PC = \coprod_{n \geq 0} \widetilde{\Sigma}_n \times_{\Sigma_n} C^{\times n}$ where $\widetilde{\Sigma}_n$ is the translation category of the symmetric group Σ_n .

Lemma 4.1. *The free functor $P: \text{Cat} \rightarrow \text{Strict}^{\text{nz}}$ sends unstable equivalences to unstable equivalences.*

Proof. This follows from the natural isomorphism of nerves $NPC \cong \coprod_{n \geq 0} N\widetilde{\Sigma}_n \times_{\Sigma_n} (NC)^{\times n}$ and the fact that $E\Sigma_n = N\widetilde{\Sigma}_n$ is a free Σ_n -space. \square

Lemma 4.2. *Let $F: X \rightarrow Y$ be an unstable (resp. stable) equivalence in $(\text{Perm}^{\text{nz}})^J$. Then*

$$\text{hocolim}_J F: \text{hocolim}_J X \rightarrow \text{hocolim}_J Y$$

is an unstable (resp. stable) equivalence. If $X: J \rightarrow \text{Perm}^{\text{nz}}$ is a constant functor and J is contractible, then

$$X(j) \rightarrow \text{hocolim}_J X$$

is an unstable equivalence.

Let I be the category of finite sets and injections and $\mathbf{m} \in I$. If $X: I \rightarrow \text{Perm}^{\text{nz}}$ is a functor such that any $\varphi: \mathbf{m} \rightarrow \mathbf{n} \in I$ is sent to an unstable (resp. stable) equivalence $X(\varphi): X(\mathbf{m}) \rightarrow X(\mathbf{n})$, then the canonical map $X(\mathbf{m}) \rightarrow \text{hocolim}_I X$ is an unstable (resp. stable) equivalence.

Proof. The stable version follows from the main theorem 4.1 in [Th3] since homotopy colimits of spectra preserve stable equivalences.

The unstable version follows from the proof of the main theorem 4.1 in [Th3]: if $F: X \rightarrow Y$ is an unstable equivalence in $(\text{Perm}^{\text{nz}})^J$, then $PF: PX \rightarrow PY$ is also an unstable equivalence by Lemma 4.1. Furthermore, there is a natural isomorphism $P \text{hocolim}_J \cong \text{hocolim}_J P$ where the leftmost hocolim is in Cat . The homotopy colimit in Cat preserves unstable equivalences, and hence $\text{hocolim}_J PX \rightarrow \text{hocolim}_J PY$ is an unstable equivalence. If X is a diagram in $\text{Strict}^{\text{nz}}$, then X has a simplicial resolution coming from the free-forgetful pair between Cat and $\text{Strict}^{\text{nz}}$. Thomason's argument [Th3, pp. 1641–1644] shows that the homotopy colimit respects this resolution and hence we get the statement for diagrams in $\text{Strict}^{\text{nz}}$. We can then extend this to general functors to Perm^{nz} as in [Th3, p. 1645].

The last statement is a weak version of Bökstedt's Lemma [Bö, 9.1] which holds for homotopy colimits in Cat since it holds for homotopy colimits in simplicial sets, and by the argument above using the resolution by free permutative categories, it also holds in Perm^{nz} . \square

4.2. The case with zero. We shall need a version of the homotopy colimit for permutative categories with zero. Thomason comments that such a homotopy colimit with zero is not a homotopy functor, unless the category is “well based”. Hence we must derive our functor to get a homotopy invariant version. One option would be to use the free-forgetful pair to resolve everything in sight by free permutative categories with zero, but since we shall be concerned with more delicate structure in our categories, we choose a less drastic approach.

The forgetful functor $U: \text{Perm} \rightarrow \text{Perm}^{\text{nz}}$ has a left adjoint $F: \text{Perm}^{\text{nz}} \rightarrow \text{Perm}$ given by $F(S) = S_+$, the category obtained by adding a disjoint zero (called “+” to distinguish it from old zeros that might live in S).

Since U and F are adjoints, we get a simplicial resolution (“monadic resolution”) Z as usual: if $S \in \text{Perm}$ and $[q] \in \Delta^{\text{op}}$ then $Z_q S = (FU)^{q+1}(S)$ with simplicial operations derived from the unit and counit of the adjunction. The counit $FU(S) \rightarrow S$ induces a map $Z(S) \rightarrow S$ of simplicial symmetric monoidal categories with zero.

Lemma 4.3. *If $S \in \text{Perm}$, then $Z(S) \rightarrow S$ is an unstable equivalence.*

Proof. The map of simplicial symmetric monoidal categories $UZ(S) \rightarrow US$ has an extra degeneracy induced by $id \rightarrow UF$. Hence the map of nerves $NZ(S) \rightarrow N(S)$ has an extra degeneracy, since the nerve only depends on the underlying category, and so it is a weak equivalence. \square

We will not define the categorical homotopy colimit on Perm , but in special cases (including all we will need) it is given in terms of the ordinary homotopy colimit.

Lemma 4.4. *If J is any small category and X and Y are functors from J to the category Perm^{nz} , then $(\text{hocolim}_J X)_+$ is functorial in transformations $f : X_+ \rightarrow Y_+$ of permutative categories with zero.*

Proof. Let \mathcal{M} be an object of Strict , which we can view as a constant functor $U\mathcal{M}$ from J to the category Perm^{nz} . The universal property of the (permutative) homotopy colimit [Th3, pp. 1626–1627] is that left lax natural transformations of functors $J \rightarrow \text{Perm}^{\text{nz}}$ from X to $U\mathcal{M}$ correspond to strict maps from $\text{hocolim}_J X$ to $U\mathcal{M}$.

The map $Y \rightarrow \text{hocolim}_J Y$ in $(\text{Perm}^{\text{nz}})^J$ induces, via the isomorphisms of adjunction

$$\begin{aligned} (\text{Perm})^J(X_+, (\text{hocolim}_J Y)_+) &\cong (\text{Perm}^{\text{nz}})^J(X, U(\text{hocolim}_J Y)_+) \\ &\cong \text{Strict}^{\text{nz}}(\text{hocolim}_J X, U(\text{hocolim}_J Y)_+) \\ &\cong \text{Strict}((\text{hocolim}_J X)_+, (\text{hocolim}_J Y)_+), \end{aligned}$$

a map

$$(\text{Perm})^J(X_+, Y_+) \rightarrow \text{Strict}((\text{hocolim}_J X)_+, (\text{hocolim}_J Y)_+).$$

Letting $\text{hocolim}_J^u f$ be the image of f , we see that if $g : Y_+ \rightarrow Z_+$ is a left lax natural transformation in $(\text{Perm})^J$ we have $\text{hocolim}_J^u g \circ \text{hocolim}_J^u f = \text{hocolim}_J^u (g \circ f)$, and we have the desired result. \square

Lemma 4.4 implies that the homotopy colimit defines a functor:

Lemma 4.5. *The assignment*

$$FX \mapsto \text{hocolim}_J^u FX := F \text{hocolim}_J X$$

defines a functor hocolim_J^u from the full subcategory of $(\text{Perm})^J$ generated by the functors $J \rightarrow \text{Perm}$ that factor through $F : \text{Perm}^{\text{nz}} \rightarrow \text{Perm}$.

The proof of Lemma 4.4 shows that this homotopy colimit has a universal property similar to the unbased homotopy colimit, and since the unbased homotopy colimit preserves (un)stable equivalences, so does hocolim^u .

Let IsolStrict denote the category of permutative categories with an isolated zero (i.e., categories in the image of F) and strict symmetric monoidal functors.

This allows us to define a derived version of the homotopy colimit with zero.

Definition 4.6. *The derived homotopy colimit*

$$D \text{hocolim}_J : (\text{Perm})^J \rightarrow \text{IsolStrict}^{\Delta^{\text{op}}}$$

is defined by

$$D \text{hocolim}_J X = \text{hocolim}_J^u ZX = \{[q] \mapsto \text{hocolim}_J^u (FU)^{q+1} X\}.$$

The construction deserves its name.

Lemma 4.7. *Let $X \rightarrow Y$ be a stable (resp. unstable) equivalence in $(\text{Perm})^J$. Then $Z_q X \rightarrow Z_q Y$ is a stable (resp. unstable) equivalence for each q , and hence the induced map*

$$D \text{hocolim}_J X \rightarrow D \text{hocolim}_J Y$$

is a stable (resp. unstable) equivalence, too.

Let \mathbf{m} be an object of the category I of finite sets and injections. If $X : I \rightarrow \text{Perm}$ is a functor such that any $\varphi : \mathbf{m} \rightarrow \mathbf{n} \in I$ is sent to an unstable (resp. stable) equivalence $X(\varphi) : X(\mathbf{m}) \rightarrow X(\mathbf{n})$, then the canonical chain

$$X(\mathbf{m}) \xleftarrow{\sim} ZX(\mathbf{m}) \longrightarrow D \text{hocolim}_I X$$

is a stable (resp. unstable) equivalence.

Remark 4.8. In the situation where S is a permutative category without zero, consider the permutative category $\mathcal{M} = S_+$. Then, for each $\mathbf{n} \in I$ the diagram $S \mapsto G\mathcal{M}(\mathbf{n}, S)$ is a diagram in $\text{Strict}^{\text{nz}}$ with a disjoint zero added, and so the based homotopy colimit over $\mathcal{Q}\mathbf{n}$ would be appropriate. However, once \mathbf{n} starts moving, the zero is used actively, and the diagram $G\mathcal{M}: I \int \mathcal{Q} \rightarrow \text{Strict}$ is in IsolStrict only, so here we will need the derived homotopy colimit.

5. THE HOMOTOPY COLIMIT OF BIPERMUTATIVE CATEGORIES

We are now ready for a key proposition:

Proposition 5.1. *Let J be a permutative category, and let \mathcal{C}^\bullet be a J -graded bipermutative category. Then $D\text{hocolim}_J \mathcal{C}^\bullet$ is a simplicial bipermutative category, and*

$$\mathcal{C}^0 \xleftarrow{\sim} Z\mathcal{C}^0 \longrightarrow D\text{hocolim}_J \mathcal{C}^\bullet$$

are maps of simplicial bipermutative categories. The same statement holds when replacing “bipermutative” by “strictly bimonoidal”.

Furthermore, for each $i \in J$, the canonical maps

$$\mathcal{C}^i \xleftarrow{\sim} Z\mathcal{C}^i \longrightarrow D\text{hocolim}_J \mathcal{C}^\bullet$$

are maps of $Z\mathcal{C}^0$ -modules.

Proof. If \mathcal{C}^\bullet is a J -graded bipermutative category, then so is FUC^\bullet , and $Z\mathcal{C}^\bullet$ becomes a simplicial J -graded bipermutative category. By Lemma 5.2 which we will prove below, we get that $\text{hocolim}_J U(FU)^q \mathcal{C}^\bullet$ becomes a zeroless bipermutative category for each q . Hence $\text{hocolim}_J^u Z_q \mathcal{C}^\bullet = F\text{hocolim}_J U(FU)^q \mathcal{C}^\bullet$ is a bipermutative category, and all the simplicial structure maps are maps of bipermutative categories. Therefore $D\text{hocolim}_J \mathcal{C}^\bullet$ becomes a simplicial bipermutative category. Likewise, for each q Lemma 5.2 below guarantees that

$$Z_q \mathcal{C}^0 \rightarrow \text{hocolim}_J^u Z_q \mathcal{C}^\bullet$$

is a map of bipermutative categories and that

$$Z_q \mathcal{C}^i \rightarrow \text{hocolim}_J^u Z_q \mathcal{C}^\bullet$$

is a map of $Z_q \mathcal{C}^0$ -modules, so we are done by functoriality. \square

Lemma 5.2. *Let J be a permutative category. If \mathcal{C}^\bullet is a J -graded bipermutative category, then Thomason’s homotopy colimit of permutative categories $\text{hocolim}_J \mathcal{C}^\bullet$ is a zeroless bipermutative category. The natural map $\mathcal{C}^0 \rightarrow \text{hocolim}_J \mathcal{C}^\bullet$ is a lax map of zeroless bipermutative categories. Furthermore, for each $i \in J$, the canonical map*

$$\mathcal{C}^i \longrightarrow \text{hocolim}_J \mathcal{C}^\bullet$$

is a map of \mathcal{C}^0 -modules.

If \mathcal{C}^\bullet is a J -graded strictly bimonoidal category, then $\text{hocolim}_J \mathcal{C}^\bullet$ is a zeroless strictly bimonoidal category with a lax map of zeroless strictly bimonoidal categories $\mathcal{C}^0 \rightarrow \text{hocolim}_J \mathcal{C}^\bullet$, and \mathcal{C}^0 -module maps $\mathcal{C}^i \rightarrow \text{hocolim}_J \mathcal{C}^\bullet$.

Proof. Thomason showed that the homotopy colimit is a permutative category without zero. There is an obvious twist map

$$\begin{aligned} \tau_\oplus: n[(x_1, X_1), \dots, (x_n, X_n)] \oplus m[(y_1, Y_1), \dots, (y_m, Y_m)] \\ \rightarrow m[(y_1, Y_1), \dots, (y_m, Y_m)] \oplus n[(x_1, X_1), \dots, (x_n, X_n)] \end{aligned}$$

that is given by $(\chi(n, m), \text{id}, \text{id})$.

For convenience we introduce the following symbolic notation: let $[X]$ be shorthand notation for $n[(x_1, X_1), \dots, (x_n, X_n)]$ and similarly $[Y]$ for $m[(y_1, Y_1), \dots, (y_m, Y_m)]$. Then we denote $[X] \oplus [Y]$ by $\begin{bmatrix} X \\ Y \end{bmatrix}$, which should be read as “first X then Y ”. The twist map τ_\oplus is then symbolically given by

$$\tau_\oplus: \begin{bmatrix} X \\ Y \end{bmatrix} \cong \begin{bmatrix} Y \\ X \end{bmatrix}.$$

In order to distinguish the multiplicative structure of \mathcal{C}^\bullet from the one on the homotopy colimit, we denote the bifunctor \otimes on \mathcal{C}^\bullet by \cdot or just by juxtaposition of objects. The multiplicative bifunctor \otimes on the homotopy colimit is then given by matrix multiplication. We define

$$\begin{aligned} n[(x_1, X_1), \dots, (x_n, X_n)] \otimes m[(y_1, Y_1), \dots, (y_m, Y_m)] \\ := nm[(x_1 + y_1, X_1 Y_1), \dots, (x_1 + y_m, X_1 Y_m), \dots, (x_n + y_1, X_n Y_1), \dots, (x_n + y_m, X_n Y_m)]. \end{aligned}$$

Again, we use shorthand notation for that and write

$$[XY] := [X] \otimes [Y].$$

The element $1 := 1[(0, 1)]$ is a unit for \otimes . With this structure $(\text{hocolim}_J \mathcal{C}^\bullet, \otimes, 1)$ is a strict monoidal category.

We define the twist map τ_\otimes for \otimes as follows, as a composite of two morphisms. Let γ_\otimes denote the twist map for the multiplication in \mathcal{C}^\bullet . First, we apply γ_\otimes on every entry of the form $X_i Y_j$. The triple $(\text{id}_{\{1, \dots, nm\}}, c_J^{x_i, y_j}, \gamma_\otimes)$ defines a morphism

$$\begin{aligned} nm[(x_1 + y_1, X_1 Y_1), \dots, (x_1 + y_m, X_1 Y_m), \dots, (x_n + y_1, X_n Y_1), \dots, (x_n + y_m, X_n Y_m)] \\ \rightarrow nm[(y_1 + x_1, Y_1 X_1), \dots, (y_m + x_1, Y_m X_1), \dots, (y_1 + x_n, Y_1 X_n), \dots, (y_m + x_n, Y_m X_n)] \end{aligned}$$

where c_J is the twist in the permutative category J . (To be precise, γ_\otimes maps $X_i Y_j$ to $(c_J^{y_j, x_i})_*(Y_j X_i)$, whereas the third coordinate of the morphism should map $(c_J^{x_i, y_j})_*(X_i Y_j)$ to $Y_j X_i$, so γ_\otimes is really an abbreviation for $(c_J^{x_i, y_j})_*(\gamma_\otimes)$.)

Second, we postcompose these maps with the morphism given by $(\sigma_{n,m}, \text{id}_{y_j + x_i}, \text{id})$, where $\sigma_{n,m} \in \Sigma_{nm}$ is the permutation that induces matrix transposition.

We write the composition $\tau_\otimes = (\sigma_{n,m}, \text{id}_{y_j + x_i}, \text{id}) \circ (\text{id}_{\{1, \dots, nm\}}, c_J^{x_i, y_j}, \gamma_\otimes)$ symbolically as

$$\tau_\otimes: [XY] \cong [YX].$$

As matrix transposition squares to the identity, $c_J^{y_j, x_i} \circ c_J^{x_i, y_j} = \text{id}$ and $\gamma_\otimes^2 = \text{id}$, we obtain that $\tau_\otimes^2 = \text{id}$. If X is the multiplicative unit, then we have that $\sigma_{1,m}$ is the identity in Σ_m and c_J^{0, y_j} is the identity as well. Similarly one shows that τ_\otimes gives the identity morphism if Y is the multiplicative unit. We leave it to the reader to check the remaining properties of 2.1 (4).

Writing out $([X] \otimes [Y]) \oplus ([X'] \otimes [Y'])$ and $([X] \oplus [X']) \otimes [Y]$ we get the same object, and we define right distributivity d_r to be the identity map between these two expressions. The left distributivity d_ℓ involves a reordering of elements. We have to have

$$d_\ell: ([X] \otimes [Y]) \oplus ([X] \otimes [Y']) = \begin{bmatrix} XY \\ XY' \end{bmatrix} \longrightarrow [X] \otimes ([Y] \oplus [Y']) = [X] \begin{bmatrix} Y \\ Y' \end{bmatrix} = [XY, XY'].$$

Here $[XY, XY']$ is shorthand notation for

$$n(m + m')[(x_1 + y_1, X_1 Y_1), \dots, (x_1 + y'_{m'}, X_1 Y'_{m'}), \dots, (x_n + y_1, X_n Y_1), \dots, (x_n + y'_{m'}, X_n Y'_{m'})].$$

The elements in the source occur in the ordering

$$(nm + nm')[(x_1 + y_1, X_1 Y_1), \dots, (x_n + y_m, X_n Y_m), (x_1 + y'_1, X_1 Y'_1), \dots, (x_n + y'_{m'}, X_n Y'_{m'})],$$

thus the source and the target do *not* agree, but they differ by a suitable permutation $\xi \in \Sigma_{nm+nm'}$. Thus we define d_ℓ as $(\xi, \text{id}, \text{id})$. Note that ξ is the left distributivity isomorphism in the bipermutative category of finite sets and surjective maps as defined in Example 2.3.

We have to check that the so defined distributivity transformation d_ℓ coincides with $\tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes)$. The twist terms γ_\otimes and c_J occur twice in the composition, so they reduce to the identity. What is left is a permutation that is caused by $\tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes)$ and this is precisely ξ .

Since the isomorphisms d_r , d_ℓ and τ_\oplus are all of the form $(\sigma, \text{id}, \text{id})$ for some permutation σ , properties (8) to (10) of Definition 2.1 follow from the ones in the bipermutative category of finite sets and surjections.

This finishes the proof that the zeroless bipermutative category structure works fine on objects.

We have to establish that \oplus and \otimes are bifunctors on $\text{hocolim}_J \mathcal{C}^\bullet$, and that the associativity and distributivity laws and the additive and multiplicative twists are natural.

For \oplus this is straightforward and can be found in [Th3]: suppose given two morphisms

$$(\psi, \ell_i, \varrho_j): n[(x_1, X_1), \dots, (x_n, X_n)] \rightarrow n'[(x'_1, X'_1), \dots, (x'_{n'}, X'_{n'})]$$

and

$$(\varphi, k_i, \pi_j): m[(y_1, Y_1), \dots, (y_m, Y_m)] \rightarrow m'[(y'_1, Y'_1), \dots, (y'_{m'}, Y'_{m'})]$$

in the homotopy colimit, with $\psi: \mathbf{n} \rightarrow \mathbf{n}'$, $\ell_i: x_i \rightarrow x'_{\psi(i)}$ and $\varrho_j: \bigoplus_{\psi(i)=j} \mathcal{C}(\ell_i)(X_i) \rightarrow X'_j$, and $\varphi: \mathbf{m} \rightarrow \mathbf{m}'$ with corresponding k_i and π_j . Then there is a surjection $\psi + \varphi$ from $\mathbf{n} + \mathbf{m}$ to $\mathbf{n}' + \mathbf{m}'$, and we can recycle the morphisms ℓ_i and k_i to give corresponding morphisms in J . In the third coordinate we can use the morphisms ϱ_j and π_j to get new ones, because the preimages of \mathbf{n}' and \mathbf{m}' under ψ and φ are disjoint. Taken together, this results in a morphism from the sum $(n+m)[(x_1, X_1), \dots, (y_m, Y_m)]$ to the sum $(n'+m')[(x'_1, X'_1), \dots, (y'_{m'}, Y'_{m'})]$. It is straightforward to see that \oplus defines a bifunctor, that the associativity law for \oplus is natural, and that the additive twist τ_{\oplus} is natural.

For the remainder of this proof let us denote the elements in the set $\mathbf{nm} = \{1, \dots, nm\}$ as pairs (i, j) with $1 \leq i \leq n$ and $1 \leq j \leq m$. The tensor product of the maps $(\psi, \ell_i, \varrho_j)$ and (φ, k_i, π_j) has three coordinates. On the first, we take the product of the surjections, *i.e.*,

$$\mathbf{nm} \ni (i, j) \mapsto (\psi(i), \varphi(j)) \in \mathbf{n}'\mathbf{m}',$$

and on the second we take the sum $\ell_i + k_j: x_i + y_j \rightarrow x'_{\psi(i)} + y'_{\varphi(j)}$ of the maps ℓ_i and k_j in J .

The third coordinate of the morphism $(\psi, \ell_i, \varrho_j) \otimes (\varphi, k_i, \pi_j)$ has to be a map

$$\bigoplus_{(\psi(i), \varphi(j))=(r,s)} \mathcal{C}(\ell_i + k_j)(X_i \cdot Y_j) = \bigoplus_{(\psi(i), \varphi(j))=(r,s)} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) \longrightarrow X'_r \cdot Y'_s.$$

Here, the sum is taken with respect to the lexicographical ordering of the indices (i, j) . Consider the following diagram.

$$\begin{array}{ccc}
& \bigoplus_{(\psi(i), \varphi(j))=(r,s)} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) & \\
& \swarrow \text{id} \quad \searrow \sigma & \\
\bigoplus_{\psi(i)=r} \bigoplus_{\varphi(j)=s} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) & & \bigoplus_{\varphi(j)=s} \bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) \\
\downarrow \bigoplus_{\psi(i)=r} d_\ell & & \downarrow \bigoplus_{\varphi(j)=s} d_r \\
\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \cdot \left(\bigoplus_{\varphi(j)=s} \mathcal{C}(k_j)(Y_j) \right) & & \bigoplus_{\varphi(j)=s} \left(\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \right) \cdot \mathcal{C}(k_j)(Y_j) \\
\downarrow \bigoplus_{\psi(i)=r} \text{id}_{\mathcal{C}(\ell_i)(X_i)} \cdot \pi_s & & \downarrow \bigoplus_{\varphi(j)=s} \varrho_r \cdot \text{id}_{\mathcal{C}(k_j)(Y_j)} \\
\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \cdot Y'_s & & \bigoplus_{\varphi(j)=s} X'_r \cdot \mathcal{C}(k_j)(Y_j) \\
\downarrow d_r & & \downarrow d_\ell \\
\left(\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \right) \cdot Y'_s & & X'_r \cdot \left(\bigoplus_{\varphi(j)=s} \mathcal{C}(k_j)(Y_j) \right) \\
\swarrow \varrho_r \otimes \text{id} \quad \searrow \text{id} \otimes \pi_s & & \\
& X'_r \cdot Y'_s &
\end{array}$$

The σ is an appropriate permutation of the summands.

The distributivity laws in \mathcal{C}^\bullet are natural with respect to morphisms in \mathcal{C}^\bullet and therefore we have that

$$\begin{aligned}
d_r \circ \left(\bigoplus_{\psi(i)=r} \text{id}_{\mathcal{C}(\ell_i)(X_i)} \cdot \pi_s \right) &= \left(\left(\text{id}_{\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i)} \right) \cdot \pi_s \right) \circ d_r, \\
d_\ell \circ \left(\bigoplus_{\varphi(j)=s} \varrho_r \cdot \text{id}_{\mathcal{C}(k_j)(Y_j)} \right) &= \left(\varrho_r \cdot \left(\text{id}_{\bigoplus_{\varphi(j)=s} \mathcal{C}(k_j)(Y_j)} \right) \right) \circ d_\ell.
\end{aligned}$$

We use the generalized pentagon equation

$$d_r \circ \bigoplus_{\psi(i)=r} d_\ell = d_\ell \circ \bigoplus_{\varphi(j)=s} d_r \circ \sigma$$

to see that the diagram commutes. We define the tensor product of the two maps to be the composition given by either of the branches.

Note that for $(\psi, \ell_i, \varrho_j) \otimes \text{id}$ the definition reduces to $(\varrho_j \cdot \text{id}) \circ d_r$, and similarly the third coordinate of $\text{id} \otimes (\varphi, k_i, \pi_j)$ is $(\text{id} \cdot \pi_j) \circ d_\ell$. In particular, the tensor product of identity morphisms is an identity morphism.

Compositions of morphisms in the homotopy colimit involve an additive twist [Th3, p. 1631]. For

$$(\psi', \ell'_i, \varrho'_j): n'[(x'_1, X'_1), \dots, (x'_{n'}, X'_{n'})] \rightarrow n''[(x''_1, X''_1), \dots, (x''_{n''}, X''_{n''})]$$

the morphism $\bigoplus_{\psi' \psi(i)=r} \mathcal{C}(\ell'_{\psi(i)} \ell_i)(X_i) \rightarrow X''_r$ is given as a composition. First, one has to permute the summands

$$\sigma: \bigoplus_{\psi' \psi(i)=r} \mathcal{C}(\ell'_{\psi(i)} \ell_i)(X_i) \rightarrow \bigoplus_{\psi'(k)=r} \bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_k \ell_i)(X_i).$$

Then, as we assumed that \mathcal{C} is a functor to Strict, we know that

$$\bigoplus_{\psi'(k)=r} \bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_k \ell_i)(X_i) = \bigoplus_{\psi'(k)=r} \bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_k) \mathcal{C}(\ell_i)(X_i) = \bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k) \left(\bigoplus_{\psi(i)=k} \mathcal{C}(\ell_i)(X_i) \right).$$

Finally, we apply the morphism

$$\bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k)(\varrho_k): \bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k) \left(\bigoplus_{\psi(i)=k} \mathcal{C}(\ell_i)(X_i) \right) \rightarrow \bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k)(X'_k)$$

and prolong this map with ϱ'_r to end up in X''_r .

In order to prove that the tensor product actually defines a bifunctor, we will show that

$$(\psi, \ell_i, \varrho_j) \otimes (\varphi, k_i, \pi_j) = ((\psi, \ell_i, \varrho_j) \otimes \text{id}) \circ (\text{id} \otimes (\varphi, k_i, \pi_j)) = (\text{id} \otimes (\varphi, k_i, \pi_j)) \circ ((\psi, \ell_i, \varrho_j) \otimes \text{id})$$

and

$$((\psi', \ell'_i, \varrho'_j) \otimes \text{id}) \circ ((\psi, \ell_i, \varrho_j) \otimes \text{id}) = ((\psi', \ell'_i, \varrho'_j) \circ (\psi, \ell_i, \varrho_j)) \otimes \text{id}$$

and leave it to the reader to check the remaining identity.

The first equation is straightforward to see, because $((\psi, \ell_i, \varrho_j) \otimes \text{id}) \circ (\text{id} \otimes (\varphi, k_i, \pi_j))$ corresponds to the left branch of the diagram above and the other composition is given by the right branch.

For the second equation we have to check that $((\varrho' \circ \varrho) \cdot \text{id}) \circ d_r = ((\varrho' \cdot \text{id}) \circ d_r \circ (\varrho \cdot \text{id}) \circ d_r)_s$. Both morphisms have source

$$\bigoplus_{\psi' \psi(i)=s} \mathcal{C}(\ell'_{\psi(i)} \ell_i + \text{id})(X_i \cdot Y_j) = \bigoplus_{\psi' \psi(i)=s} \mathcal{C}(\ell'_{\psi(i)} \ell_i)(X_i) \cdot Y_j$$

and the left hand side corresponds to the left branch of the following diagram and the right hand side to the right branch.

$$\begin{array}{ccc} \bigoplus_{\psi' \psi(i)=s} \mathcal{C}(\ell'_{\psi(i)} \ell_i)(X_i) \cdot Y_j & \xlongequal{\quad} & \bigoplus_{\psi' \psi(i)=s} \mathcal{C}(\ell'_{\psi(i)} \ell_i + \text{id})(X_i \cdot Y_j) \\ \downarrow d_r & & \downarrow \sigma \\ \left(\bigoplus_{\psi' \psi(i)=s} \mathcal{C}(\ell'_{\psi(i)} \ell_i)(X_i) \right) \cdot Y_j & & \bigoplus_{\psi'(k)=s} \bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_{\psi(i)} + \text{id}) \mathcal{C}(\ell_i + \text{id})(X_i \cdot Y_j) \\ \downarrow \sigma \cdot \text{id} & & \parallel \\ \left(\bigoplus_{\psi'(k)=s} \bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_{\psi(i)} \ell_i) \mathcal{C}(\ell_i)(X_i) \right) \cdot Y_j & & \bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k + \text{id}) \left(\bigoplus_{\psi(i)=k} \mathcal{C}(\ell_i + \text{id})(X_i \cdot Y_j) \right) \\ \parallel & & \downarrow \bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k + \text{id})((\varrho_k \cdot \text{id}) \circ d_r) \\ \left(\bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k) \left(\bigoplus_{\psi(i)=k} \mathcal{C}(\ell_i)(X_i) \right) \right) \cdot Y_j & & \bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k)(X'_k) \cdot Y_j \\ \downarrow (\bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k)(\varrho_k)) \cdot \text{id} & & \downarrow d_r \\ \left(\bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k)(X'_k) \right) \cdot Y_j & \xlongequal{\quad} & \left(\bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k)(X'_k) \right) \cdot Y_j \\ & \searrow \varrho'_s \cdot \text{id} \quad \swarrow \varrho'_s \cdot \text{id} & \\ & X''_s \cdot Y_j & \end{array}$$

Naturality of d_r in \mathcal{C}^\bullet ensures that d_r can change place with $\bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k + \text{id})(\varrho_k \cdot \text{id})$ on the right branch. That $d_r \circ \sigma = (\sigma \cdot \text{id}) \circ d_r$ holds because \mathcal{C}^\bullet satisfies property (8) from Definition 2.1 and hence the diagram commutes.

In order to show that the associativity identification is natural, we have to prove that

$$((\psi^1, \ell_i^1, \varrho_j^1) \otimes (\psi^2, \ell_i^2, \varrho_j^2)) \otimes (\psi^3, \ell_i^3, \varrho_j^3) = (\psi^1, \ell_i^1, \varrho_j^1) \otimes ((\psi^2, \ell_i^2, \varrho_j^2) \otimes (\psi^3, \ell_i^3, \varrho_j^3))$$

for morphisms in the homotopy colimit. The claim is obvious on the coordinates of the surjections and the morphisms in J .

For proving the identity in the third coordinate of morphisms, note that the naturality of \otimes implies that we can write

$$\begin{aligned} & ((\psi^1, \ell_i^1, \varrho_j^1) \otimes (\psi^2, \ell_i^2, \varrho_j^2)) \otimes (\psi^3, \ell_i^3, \varrho_j^3) \\ &= (((\psi^1, \ell_i^1, \varrho_j^1) \otimes \text{id}) \otimes \text{id}) \circ ((\text{id} \otimes (\psi^2, \ell_i^2, \varrho_j^2)) \otimes \text{id}) \circ ((\text{id} \otimes \text{id}) \otimes (\psi^3, \ell_i^3, \varrho_j^3)). \end{aligned}$$

Therefore, it suffices to prove the claim for each of the factors. We will show it for the middle one and leave the other ones to the curious reader. Recall that $\text{id} \otimes (\psi^2, \ell_i^2, \varrho_j^2)$ has as third coordinate the composition $(\text{id} \cdot \varrho_j^2) \circ d_\ell$ and therefore $(\text{id} \otimes (\psi^2, \ell_i^2, \varrho_j^2)) \otimes \text{id}$ has third coordinate

$$(((\text{id} \cdot \varrho_j^2) \circ d_\ell) \cdot \text{id}) \circ d_r = (\text{id} \cdot \varrho_j^2 \cdot \text{id}) \circ (\text{id} \cdot d_\ell) \circ d_r.$$

But $(\text{id} \cdot d_\ell) \circ d_r = (d_r \cdot \text{id}) \circ d_\ell$ (equation (7') of Definition 2.4) holds in \mathcal{C}^\bullet , and therefore the third coordinate equals

$$(\text{id} \cdot \varrho_j^2 \cdot \text{id}) \circ (\text{id} \cdot d_\ell) \circ d_r = (\text{id} \cdot \varrho_j^2 \cdot \text{id}) \circ (d_r \cdot \text{id}) \circ d_\ell$$

which is the third coordinate of $\text{id} \otimes ((\psi^2, \ell_i^2, \varrho_j^2) \otimes \text{id})$.

Naturality of the multiplicative twist map can be seen as follows. We have to show that

$$\tau_\otimes \circ ((\psi, \ell_i, \varrho_j) \otimes (\varphi, k_i, \pi_j)) = ((\psi, \ell_i, \varrho_j) \otimes (\varphi, k_i, \pi_j)) \circ \tau_\otimes.$$

On the first coordinate of the morphisms this reduces to the equality

$$\sigma_{n', m'} \circ (\psi, \varphi)(i, j) = (\varphi(j), \psi(i)) = (\varphi, \psi) \circ \sigma_{n, m}(i, j),$$

and on the second coordinate we have the equation

$$c_J \circ (\ell_i + k_j) = (k_j + \ell_i) \circ c_J$$

because c_J is natural. Thus, it remains to prove that the above equation holds in the third coordinate, which amounts to showing that the following diagram commutes.

$$\begin{array}{ccc} \bigoplus_{\psi(i)=r} \bigoplus_{\varphi(j)=s} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) & \xrightarrow{(\bigoplus \bigoplus \gamma_\otimes) \circ \sigma} & \bigoplus_{\varphi(j)=s} \bigoplus_{\psi(i)=r} \mathcal{C}(k_j)(Y_j) \cdot \mathcal{C}(\ell_i)(X_i) \\ \downarrow \bigoplus_{\psi(i)=r} d_\ell & & \downarrow \sigma^{-1} \\ \bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \cdot \left(\bigoplus_{\varphi(j)=s} \mathcal{C}(k_j)(Y_j) \right) & \xrightarrow{\bigoplus_{\psi(i)=r} \gamma_\otimes} & \bigoplus_{\psi(i)=r} \left(\bigoplus_{\varphi(j)=s} \mathcal{C}(k_j)(Y_j) \right) \cdot \mathcal{C}(\ell_i)(X_i) \\ \downarrow \bigoplus_{\psi(i)=r} \text{id} \cdot \pi_s & & \downarrow \bigoplus_{\psi(i)=r} d_r \\ \bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \cdot Y'_s & & \bigoplus_{\psi(i)=r} Y'_s \cdot \mathcal{C}(\ell_i)(X_i) \\ \downarrow d_r & & \downarrow \bigoplus_{\psi(i)=r} \pi_s \cdot \text{id} \\ \left(\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \right) \cdot Y'_s & & Y'_s \cdot \left(\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \right) \\ \downarrow \varrho_r \cdot \text{id} & & \downarrow d_\ell \\ X'_r \cdot Y'_s & \xrightarrow{\gamma_\otimes} & Y'_s \cdot X'_r \end{array}$$

The top diagram commutes because d_ℓ is defined in terms of d_r and γ_\otimes . For the bottom diagram we apply the same argument together with the naturality of γ_\otimes .

We have to check that right distributivity is the identity on morphisms. Consider three morphisms as above. When we focus on the surjections $\psi^1: n \rightarrow n'$, $\psi^2: m \rightarrow m'$, and $\psi^3: \ell \rightarrow \ell'$, we see that a condition like $(\psi^1 + \psi^2)\psi^3(i, j) = (r, s)$ only affects either the preimage of $n'\ell'$ or the preimage of $m'\ell'$ in $(n + m)\ell$, but never both. Therefore, the third coordinate of the morphism

$$((\psi^1, \ell_i^1, \varrho_j^1) \oplus (\psi^2, \ell_i^2, \varrho_j^2)) \otimes (\psi^3, \ell_i^3, \varrho_j^3)$$

is either a third coordinate of $(\psi^1, \ell_i^1, \varrho_j^1) \otimes (\psi^3, \ell_i^3, \varrho_j^3)$ or of $(\psi^2, \ell_i^2, \varrho_j^2) \otimes (\psi^3, \ell_i^3, \varrho_j^3)$ and thus right distributivity is the identity on morphisms.

In the J -graded bipermutative case the naturality of the left distributivity isomorphism follows from the one of d_r and the multiplicative twist. In the bipermutative and the strictly bimonoidal case left distributivity is given by $(\xi, \text{id}, \text{id})$. Therefore naturality of d_ℓ in the bipermutative setting proves naturality in the strictly bimonoidal setting as well.

This finishes the proof that the homotopy colimit $\text{hocolim}_J \mathcal{C}^\bullet$ is a bipermutative category without zero.

We now prove the remaining statements of the lemma. There is a natural functor G from \mathcal{C}^0 to $\text{hocolim}_J \mathcal{C}^\bullet$ which sends $X \in \mathcal{C}^0$ to $1[(0, X)]$. Note that the functor G is strict (symmetric) monoidal with respect to \otimes , because $G(1) = 1[(0, 1)]$ and

$$G(X) \otimes G(Y) = 1[(0, X)] \otimes 1[(0, Y)] = 1[(0 + 0, X \otimes Y)] = 1[(0, X \otimes Y)] = G(X \otimes Y).$$

However, G is only lax symmetric monoidal with respect to \oplus : there is a binatural morphism η_\oplus from $G(X) \oplus G(Y) = 1[(0, X)] \oplus 1[(0, Y)] = 2[(0, X), (0, Y)]$ to $G(X \oplus Y) = 1[(0, X \oplus Y)]$ given by the canonical surjection ψ from $\mathbf{2}$ to $\mathbf{1}$ and identity morphisms in the other two components, but of course this map is not an isomorphism. We have to show that the functor G respects the distributivity constraints $d_r = \text{id}$ and d_ℓ . In our situation we have that $\eta_\otimes = \text{id}$, thus we have to check that

$$\eta_\oplus = \eta_\oplus \otimes \text{id}$$

and

$$(\text{id} \otimes \eta_\oplus) \circ \tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes) = G(\tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes)) \circ \eta_\oplus.$$

The first equation is just stating the fact that

$$\begin{array}{ccc} 2[(0, XY), (0, X'Y)] & \xlongequal{\quad} & 2[(0, X), (0, X')] \otimes 1[(0, Y)] \\ \eta_\oplus = (\psi, \text{id}, \text{id}) \downarrow & & \downarrow \eta_\oplus \otimes \text{id} = (\psi, \text{id}, \text{id}) \otimes \text{id} \\ 1[0, XY \oplus X'Y] & \xlongequal{\quad} & 1[0, (X \oplus X')Y] \end{array}$$

commutes.

For the left distributivity law we should observe that the multiplicative twist τ_\otimes on the homotopy colimit reduces to the morphism $(\text{id}, c_J, \gamma_\otimes)$ in the case of elements of length 1 in the homotopy colimit, and that $c_J^{0,0} = \text{id}$. Furthermore, $\text{id} \otimes (\psi, \text{id}, \text{id}) = (\psi, \text{id}, \text{id})$ holds. Therefore

$$\begin{aligned} (\text{id} \otimes \eta_\oplus) \circ d_\ell &= (\text{id} \otimes (\psi, \text{id}, \text{id})) \circ \tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes) \\ &= (\psi, \text{id}, \text{id}) \circ (\text{id}, \text{id}, \gamma_\otimes \circ (\gamma_\otimes \oplus \gamma_\otimes)) \\ &= (\text{id}, \text{id}, \gamma_\otimes \circ (\gamma_\otimes \oplus \gamma_\otimes)) \circ (\psi, \text{id}, \text{id}) = G(d_\ell) \circ \eta_\oplus. \end{aligned}$$

The claim about the module structure is obvious.

As the left distributivity on the homotopy colimit is of the form $(\xi, \text{id}, \text{id})$, the above proof carries over to the strictly bimonoidal case. \square

Lemma 5.3. *If $g: \mathcal{C}^\bullet \rightarrow \mathcal{D}^\bullet$ is a lax morphism of J -graded bipermutative categories (resp. J -graded strictly bimonoidal categories) then it induces a lax morphism of zeroless bipermutative categories (resp. zeroless strictly bimonoidal categories) $g_*: \text{hocolim}_J \mathcal{C}^\bullet \rightarrow \text{hocolim}_J \mathcal{D}^\bullet$.*

Proof. Of course, we define $g_*: \text{hocolim}_J \mathcal{C}^\bullet \rightarrow \text{hocolim}_J \mathcal{D}^\bullet$ as

$$g_*(n[(x_1, A_1), \dots, (x_n, A_n)]) := n[(x_1, g(A_1)), \dots, (x_n, g(A_n))].$$

Note that with this definition g_* is *strict* symmetric monoidal with respect to \oplus even if g was only lax symmetric monoidal.

For a morphism $(\psi, \ell_i, \varrho_j)$ from $n[(x_1, A_1), \dots, (x_n, A_n)]$ to $m[(y_1, B_1), \dots, (y_m, B_m)]$ we define an induced morphism

$$g_*(n[(x_1, A_1), \dots, (x_n, A_n)]) \rightarrow g_*(m[(y_1, B_1), \dots, (y_m, B_m)])$$

as follows: we keep the surjection ψ and the maps ℓ_i . For

$$\varrho_j: \bigoplus_{\psi(i)=j} A_i \rightarrow B_j$$

we take the composition

$$\varrho_j^g: \bigoplus_{\psi(i)=j} g(A_i) \xrightarrow{\eta_{\oplus}} g\left(\bigoplus_{\psi(i)=j} A_i\right) \xrightarrow{g(\varrho_j)} g(B_j)$$

and obtain a morphism $(\psi, \ell_i, \varrho_j^g)$ on the homotopy colimit. The naturality of η_{\oplus} ensures that composition of morphisms is well-defined.

Let $n[(x_1, A_1), \dots, (x_n, A_n)]$ and $m[(y_1, B_1), \dots, (y_m, B_m)]$ be two objects in $\mathrm{hocolim}_J \mathcal{C}^\bullet$. Applying $g_* \circ (- \otimes -)$ yields

$$nm[(x_1 + y_1, g(A_1 \otimes B_1)), \dots, (x_n + y_m, g(A_n \otimes B_m))]$$

whereas the composition $(- \otimes -) \circ (g_*, g_*)$ gives

$$nm[(x_1 + y_1, g(A_1) \otimes g(B_1)), \dots, (x_n + y_m, g(A_n) \otimes g(B_m))].$$

Thus, we can use $(\mathrm{id}, \mathrm{id}, \eta_{\otimes})$ to obtain a natural transformation η_{\otimes}^h from $(- \otimes -) \circ (g_*, g_*)$ to $g_* \circ (- \otimes -)$. This transformation inherits all properties from η_{\otimes} , in particular, η_{\otimes}^h is lax symmetric monoidal if η_{\otimes} was so.

It remains to check the properties concerning the distributivity laws. As d_r is the identity on the J -graded bipermutative category and on the homotopy colimit, and η_{\oplus} is the identity on the homotopy colimit, the equalities reduce to

$$(1) \quad (\eta_{\otimes}^h \oplus \eta_{\otimes}^h) = \eta_{\otimes}^h$$

and

$$(2) \quad g_*(d_\ell) \circ (\eta_{\otimes}^h \oplus \eta_{\otimes}^h) = \eta_{\otimes}^h \circ d_\ell.$$

The first equation is straightforward to check.

The left distributivity law in the homotopy colimit is given by $d_\ell = (\xi, \mathrm{id}, \mathrm{id})$ and $\eta_{\otimes}^h \oplus \eta_{\otimes}^h$ is equal to

$$\eta_{\otimes}^h \oplus \eta_{\otimes}^h = (\mathrm{id}_{\mathbf{nm}}, \mathrm{id}_{x_i+y_j}, \eta_{\otimes}) \oplus (\mathrm{id}_{\mathbf{nm}'}, \mathrm{id}_{x_i+z_k}, \eta_{\otimes}).$$

As addition in the homotopy colimit is given by concatenation, this shows that we can simplify the above expression to $(\mathrm{id}_{\mathbf{nm}+\mathbf{nm}'}, \mathrm{id}_{x_i+u_r}, \eta_{\otimes})$ where u_r is either of the form y_j or z_k . As d_ℓ differs from the identity only in the first coordinate, and $\eta_{\otimes}^h \oplus \eta_{\otimes}^h$ only in the third coordinate, these maps commute. \square

6. A MULTIPLICATIVE GROUP COMPLETION DEVICE

Recall from 3.2 the construction $GM: I \int \mathcal{Q} \rightarrow \mathrm{Strict}$.

Lemma 6.1. *Let \mathcal{M} be a permutative category. Then*

- (1) *the canonical map $\mathcal{M} \rightarrow \mathrm{hocolim}_{I \int \mathcal{Q}} GM$ is a stable equivalence,*
- (2) *the canonical map $\mathrm{hocolim}_{S \in \mathcal{Q} \mathbf{1}} GM(\mathbf{1}, S) \rightarrow \mathrm{hocolim}_{I \int \mathcal{Q}} GM$ is an unstable equivalence and*
- (3) *$\mathrm{hocolim}_{I \int \mathcal{Q}} GM$ is group complete.*

Proof. Recall that spectrification commutes with homotopy colimits, i.e., $\mathrm{hocolim}_J \mathrm{Spt}$ is equivalent to $\mathrm{Spt} \mathrm{hocolim}_J$. Given $\mathbf{n} \in I$, the homotopy colimit $\mathrm{hocolim}_{S \in \mathcal{Q} \mathbf{n}} \mathrm{Spt} GM(\mathbf{n}, S)$ can be calculated by taking the homotopy colimit in each direction of $\mathcal{Q} \mathbf{n}$ successively. Since all nontrivial maps involved are diagonal maps, we see that the homotopy colimit in the k th direction can be identified with the homotopy colimit $\mathrm{hocolim}_{S \in \mathcal{Q}(\mathbf{n}-\mathbf{1})} \mathrm{Spt} GM((\mathbf{n}-\mathbf{1}), S)$ through the ordered injection $(\mathbf{n}-\mathbf{1}) \rightarrow \mathbf{n}$ which skips k . Hence, the I -diagram $\mathbf{n} \mapsto \mathrm{hocolim}_{S \in \mathcal{Q} \mathbf{n}} \mathcal{M}$ is stably constant. The last part of Lemma 4.2 then says that the map $\mathcal{M} \rightarrow \mathrm{hocolim}_{\mathbf{n} \in I} \mathrm{hocolim}_{S \in \mathcal{Q} \mathbf{n}} GM(\mathbf{n}, S)$ is a stable equivalence.

The claim that the map $\mathcal{M} \rightarrow \mathrm{hocolim}_{I \int \mathcal{Q}} GM$ is a stable equivalence follows, since by extending Thomason's proof [Th1] of $\mathrm{hocolim}_I \mathcal{Q} \simeq N(I \int \mathcal{Q}) = \mathrm{hocolim}_{I \int \mathcal{Q}} *$ (for the trivial functor $*$) to allow for arbitrary functors from $I \int \mathcal{Q}$, we have an equivalence

$$\mathrm{hocolim}_{I \int \mathcal{Q}} \mathrm{Spt} GM \simeq \mathrm{hocolim}_{\mathbf{n} \in I} \mathrm{hocolim}_{S \in \mathcal{Q} \mathbf{n}} \mathrm{Spt} GM(\mathbf{n}, S).$$

See also [Sch, theorem 2.3] for a write-up in the dual situation.

That π_0 is a group is seen as follows. It is enough to show that elements of the form $1[(\mathbf{n}, S), a]$ have negatives.

If $S \neq \mathbf{n}_+$ there is a map $S \subseteq T \in \mathcal{Q}\mathbf{n}$ with T containing a negative number, so there is a path $1[(\mathbf{n}, S), a] \rightarrow 1[(\mathbf{n}, T), 0] \leftarrow 1[(\mathbf{0}, \{0\}), 0]$, and our element is zero. If $S = \mathbf{n}_+$, let $i: \mathbf{n} \rightarrow \mathbf{n} + \mathbf{1}$ be the inclusion skipping $n + 1$, and let $a' \in \mathcal{M}^{\mathbf{n}+1}$ be defined by $a'_U = a_{U - \{n+1\}}$ if $n + 1 \in U$ and $a'_U = 0$ otherwise, so that $(\mathcal{M}_i a) + a' = \mathcal{M}_{\mathbf{n}+1} i(a)$. Then $1[(\mathbf{n}, \mathbf{n}_+), a] \rightarrow 1[(\mathbf{n} + \mathbf{1}, (\mathbf{n} + \mathbf{1})_+), \mathcal{M}_i a]$ is a path and

$$\begin{aligned} & 1[(\mathbf{n} + \mathbf{1}, (\mathbf{n} + \mathbf{1})_+), \mathcal{M}_i a] + 1[(\mathbf{n} + \mathbf{1}, (\mathbf{n} + \mathbf{1})_+), a'] \\ &= 2[(\mathbf{n} + \mathbf{1}, (\mathbf{n} + \mathbf{1})_+), \mathcal{M}_i a, ((\mathbf{n} + \mathbf{1}, (\mathbf{n} + \mathbf{1})_+), a')] \\ &\rightarrow 1[(\mathbf{n} + \mathbf{1}, (\mathbf{n} + \mathbf{1})_+), (\mathcal{M}_i a) + a'] \\ &= 1[(\mathbf{n} + \mathbf{1}, (\mathbf{n} + \mathbf{1})_+), \mathcal{M}_{\mathbf{n}+1} i(a)] \leftarrow 1[(\mathbf{n} + \mathbf{1}, \mathbf{n}_+), a], \end{aligned}$$

and the latter is, as we saw above, in the same component as zero.

Now, since stable equivalences between group complete symmetric monoidal categories are unstable equivalences, the second claim also follows. \square

Lemma 6.2. *If \mathcal{M} is a permutative category with zero with all morphisms isomorphisms and additive translation is faithful, then there is an unstable equivalence*

$$\mathrm{hocolim}_{S \in \mathcal{Q}\mathbf{1}} G\mathcal{M}(\mathbf{1}, S) \rightarrow (-\mathcal{M})\mathcal{M}.$$

Proof. This is entirely due to Thomason [Th3]. Theorem 5.2 in *loc. cit.* asserts that the map from $\mathrm{hocolim}_{S \in \mathcal{Q}\mathbf{1}} G\mathcal{M}(\mathbf{1}, S)$ to the “simplified mapping cylinder” is an unstable equivalence and his argument on pages 1649 to 1658 shows that a map from the simplified mapping cylinder to $(-\mathcal{M})\mathcal{M}$ is an unstable equivalence. \square

The map $\mathrm{hocolim}_{S \in \mathcal{Q}\mathbf{1}} G\mathcal{M}(\mathbf{1}, S) \rightarrow (-\mathcal{M})\mathcal{M}$ which is proved to be an unstable equivalence in Lemma 6.2 is the additive extension of the assignment which sends both $1[\{-1, 0\}, 0]$ and $1[\{0\}, a]$ to $(0, 0) \in (-\mathcal{M})\mathcal{M}$ and $1[\{0, 1\}, (a, b)]$ to (a, b) . The map on morphisms is straightforward, once one declares that the morphism $1[\{0\}, a] \rightarrow 1[\{0, 1\}, (a, a)]$ is sent to $[id_a, a]: (0, 0) \rightarrow (a, a) \in (-\mathcal{M})\mathcal{M}$.

Collecting the results, we obtain our multiplicative group completion.

Corollary 6.3. *Let \mathcal{R} be a bipermutative category or a strictly bimonoidal category. The canonical map $\mathcal{R} \rightarrow \mathrm{hocolim}_{I \downarrow \mathcal{Q}} G\mathcal{R}$ is a map of bipermutative categories or strictly bimonoidal categories without zero, and $\mathrm{hocolim}_{S \in \mathcal{Q}\mathbf{1}} G\mathcal{R}(\mathbf{1}, S) \rightarrow \mathrm{hocolim}_{I \downarrow \mathcal{Q}} G\mathcal{R}$ is an unstable equivalence.*

Adding units and tracing the action of $Z\mathcal{R}$, we have the main result:

Theorem 6.4. *If \mathcal{R} is a commutative rig category (or a rig category), then*

$$\bar{\mathcal{R}} = D \mathrm{hocolim}_{I \downarrow \mathcal{Q}} G\mathcal{R}$$

is a simplicial commutative ring category (resp. a simplicial ring category), where $G\mathcal{R}$ is the $I \downarrow \mathcal{Q}$ -graded bipermutative category (resp. $I \downarrow \mathcal{Q}$ -graded strictly bimonoidal category) of Proposition 3.2 applied to the bipermutative category (resp. strictly bimonoidal category) associated with \mathcal{R} .

The rig maps of Proposition 5.1

$$\mathcal{R} \xleftarrow{\sim} Z\mathcal{R} \longrightarrow \bar{\mathcal{R}}$$

are stable equivalences. Furthermore, if \mathcal{R} is a groupoid with faithful additive translation, then the maps

$$(-\mathcal{R})\mathcal{R} \xleftarrow{\sim} Z(-\mathcal{R})\mathcal{R} \xleftarrow{\sim} Z \mathrm{hocolim}_{S \in \mathcal{Q}\mathbf{1}} G\mathcal{R}(\mathbf{1}, S) \xrightarrow{\sim} \bar{\mathcal{R}}$$

form a chain of unstable equivalences of $Z\mathcal{R}$ -modules.

7. APPENDIX: AN ALTERNATIVE CONSTRUCTION

We sketch an alternative construction of a group completion. First, let us recall a slightly modified version of the Elmendorf–Mandell model of K -theory. There is a precursor of this model in Shimakawa’s papers [Sh, pp. 378–379]. Let $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, c_{\oplus}, \otimes, 1_{\mathcal{R}})$ be a small strictly bimonoidal category. For now we focus on the additive structure $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, c_{\oplus})$ of \mathcal{R} . The following is taken from [EM, §4].

For finite based sets X_+^1, \dots, X_+^n with $+$ denoting the basepoint, $\bar{H}\mathcal{R}(X_+^1, \dots, X_+^n)$ is the category with objects $(C_{\langle S \rangle}, \rho(\langle S \rangle; i, T, U))$ where

- $\langle S \rangle = (S_1, \dots, S_n)$ is an n -tuple of basepoint-free subsets $S_i \subseteq X^i$.
- The $C_{\langle S \rangle}$ are objects of \mathcal{R} .

- Let $\langle S; i, T \rangle$ denote $(S_1, \dots, S_{i-1}, T, S_{i+1}, \dots, S_n)$ for some subset $T \subseteq S_i$. Then the $\rho(\langle S \rangle; i, T, U)$ are isomorphisms from $C_{\langle S; i, T \rangle} \oplus C_{\langle S; i, U \rangle}$ to $C_{\langle S \rangle}$ for $i = 1, \dots, n$ and $T, U \subseteq S_i$ with $T \cap U = \emptyset$ and $T \cup U = S_i$.

The $(C_{\langle S \rangle}, \rho(\langle S \rangle; i, T, U))$ satisfy the following properties.

- (1) If one $S_i = \emptyset$ for $i \in \{1, \dots, n\}$, then $C_{\langle S \rangle} = 0_{\mathcal{R}}$.
- (2) If one of the S_i or T or U is empty, then $\rho(\langle S \rangle; i, T, U) = \text{id}$.
- (3) If c_{\oplus} denotes the twist of the permutative structure $(\mathcal{R}, \oplus, 0_{\mathcal{R}})$, then

$$\rho(\langle S \rangle; i, T, U) = \rho(\langle S \rangle; i, U, T) \circ c_{\oplus}.$$

- (4) The $\rho(\langle S \rangle; i, T, U)$ are associative, i.e., for all $\langle S \rangle, i$ and pairwise disjoint $T, U, V \subseteq S_i$ with $T \cup U \cup V = S_i$ the diagram

$$\begin{array}{ccc} C_{\langle S; i, T \rangle} \oplus C_{\langle S; i, U \rangle} \oplus C_{\langle S; i, V \rangle} & \xrightarrow{\rho(\langle S; i, T \cup U \rangle; i, T, U) \oplus \text{id}} & C_{\langle S; i, T \cup U \rangle} \oplus C_{\langle S; i, V \rangle} \\ \text{id} \oplus \rho(\langle S; i, U \cup V \rangle; i, U, V) \downarrow & & \downarrow \rho(\langle S \rangle; i, T \cup U, V) \\ C_{\langle S; i, T \rangle} \oplus C_{\langle S; i, U \cup V \rangle} & \xrightarrow{\rho(\langle S \rangle; i, T, U \cup V)} & C_{\langle S \rangle} \end{array}$$

commutes.

- (5) The $\rho(\langle S \rangle; i, T, U)$ satisfy the pentagon rule, i.e., for $i \neq j$ and $T, U \subseteq S_i$, $V, W \subseteq S_j$ with $T \cap U = \emptyset = V \cap W$ the diagram

$$\begin{array}{ccc} & & C_{\langle S; j, V \rangle} \oplus C_{\langle S; j, W \rangle} \\ & \nearrow \rho(\langle S; j, V \rangle; i, T, U) \oplus \rho(\langle S; j, W \rangle; i, T, U) & \\ C_{\langle S; i, T; j, V \rangle} \oplus C_{\langle S; i, U; j, V \rangle} \oplus C_{\langle S; i, T; j, W \rangle} \oplus C_{\langle S; i, U; j, W \rangle} & & \\ \text{id} \oplus c_{\oplus} \oplus \text{id} \downarrow & & \downarrow \rho(\langle S \rangle; j, V, W) \\ C_{\langle S; i, T; j, V \rangle} \oplus C_{\langle S; i, T; j, W \rangle} \oplus C_{\langle S; i, U; j, V \rangle} \oplus C_{\langle S; i, U; j, W \rangle} & & C_{\langle S \rangle} \\ \rho(\langle S; i, T \rangle; j, V, W) \oplus \rho(\langle S; i, U \rangle; j, V, W) \searrow & & \nearrow \rho(\langle S \rangle; i, T, U) \\ & C_{\langle S; i, T \rangle} \oplus C_{\langle S; i, U \rangle} & \end{array}$$

commutes.

Morphisms in the category consist of morphisms $f_{\langle S \rangle}: C_{\langle S \rangle} \rightarrow D_{\langle S \rangle}$ in \mathcal{R} that are the identity if any of the S_i is empty. These morphisms have to commute with the structure maps $\rho(\langle S \rangle; i, T, U)$.

Thus $\bar{H}\mathcal{R}$ is a functor from the n -fold product of the category Γ of finite pointed sets to the category of permutative categories. If $f: X_+ \rightarrow Y_+$ is a map of finite pointed sets and $(C_{\langle S \rangle}, \rho(\langle S \rangle; i, T, U))$ is an object in $\bar{H}\mathcal{R}$, then $f_*(C_{\langle S \rangle}, \rho(\langle S \rangle; i, T, U))$ is the object that is given by the cube with values $f_*C_{\langle S \rangle} := C_{\langle f^{-1}(S) \rangle}$ for all subsets S of Y . As f respects the basepoint $+$, this is well-defined because $f^{-1}(T)$ does not contain the basepoint for $T \subseteq S$. The structure maps $f_*\rho$ are given by

$$\begin{array}{ccc} f_*C_{\langle S; i, T \rangle} \oplus f_*C_{\langle S; i, U \rangle} & \xlongequal{\quad} & C_{\langle f^{-1}S; i, f^{-1}T \rangle} \oplus C_{\langle f^{-1}S; i, f^{-1}U \rangle} \\ & & \downarrow \rho(\langle f^{-1}S \rangle; i, f^{-1}T, f^{-1}U) \\ & & C_{\langle f^{-1}S; i, f^{-1}T \cup f^{-1}U \rangle} = f_*C_{\langle S \rangle}. \end{array}$$

The permutative structure on $\bar{H}\mathcal{R}$ is given by sending $(C_{\langle S \rangle}, \rho(\langle S \rangle; i, T, U))$ and $(D_{\langle S \rangle}, \rho'(\langle S \rangle; i, T, U))$ to the object

$$(C_{\langle S \rangle} \oplus D_{\langle S \rangle}, \rho''(\langle S \rangle; i, T, U))$$

where the structure map ρ'' is given by

$$\begin{array}{ccc}
(C \oplus D)_{\langle S; i, T \rangle} \oplus (C \oplus D)_{\langle S; i, U \rangle} & \xlongequal{\quad} & C_{\langle S; i, T \rangle} \oplus D_{\langle S; i, T \rangle} \oplus C_{\langle S; i, U \rangle} \oplus D_{\langle S; i, U \rangle} \\
& & \downarrow \text{id} \oplus c_{\oplus} \oplus \text{id} \\
& & C_{\langle S; i, T \rangle} \oplus C_{\langle S; i, U \rangle} \oplus D_{\langle S; i, T \rangle} \oplus D_{\langle S; i, U \rangle} \\
& & \downarrow \rho(\langle S \rangle; i, T, U) \oplus \rho'(\langle S \rangle; i, T, U) \\
& & C_{\langle S \rangle} \oplus D_{\langle S \rangle}.
\end{array}$$

The above definition is quite close to the one in [EM]; however, we require the gluing maps ρ to be isomorphisms.

The case $n = 1$ is well studied: let m_+ denote the finite pointed set $\{0, 1, \dots, m\}$ with 0 as basepoint.

Lemma 7.1. [ShSh, Lemma 2.2] *The canonical map*

$$\bar{H}\mathcal{R}(m_+) \longrightarrow \bar{H}\mathcal{R}(1_+) \times \dots \times \bar{H}\mathcal{R}(1_+)$$

is an equivalence of categories.

Let X_1, \dots, X_n be finite pointed simplicial sets. We define $\bar{H}\mathcal{R}(X_1, \dots, X_n)$ to be the n -simplicial permutative category with

$$\bar{H}\mathcal{R}(X_1, \dots, X_n)_{(\ell_1, \dots, \ell_n)} := \bar{H}\mathcal{R}((X_1)_{\ell_1}, \dots, (X_n)_{\ell_n})$$

for $\ell_i \in \Delta$.

Definition 7.2. For a strictly bimonoidal category \mathcal{R} we define $K^n\mathcal{R}$ to be the category that is the limit of the diagram

$$\bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n})$$

with $i_j \in \{0, 1, 2\}$ and $Y_0 = \mathbb{S}^1$, $Y_1 = Y_2 = P\mathbb{S}^1$ and $d_0: P\mathbb{S}^1 \rightarrow \mathbb{S}^1$.

Here, \mathbb{S}^1 is the small simplicial model of the 1-sphere, P denotes the simplicial model of the path space functor which takes a simplicial set X to the simplicial set PX with $PX_n = X_{n+1}$. The map $d_0: X_{n+1} \rightarrow X_n$ induces a map $PX \rightarrow X$.

More generally, we define for pointed simplicial sets X_1, \dots, X_n

$$\tilde{H}\mathcal{R}(X_1, \dots, X_n) := \lim \bar{H}\mathcal{R}(Y_{i_1} \wedge X_1, \dots, Y_{i_n} \wedge X_n)$$

where the Y_{i_j} are as above.

Note that $K^1\mathcal{R}$ corresponds to the classical case ([ShSh, M3, Se]). It is the pullback of the diagram

$$\begin{array}{ccc}
& \bar{H}\mathcal{R}(P\mathbb{S}^1) & \\
& \downarrow d_0 & \\
\bar{H}\mathcal{R}(P\mathbb{S}^1) & \xrightarrow{d_0} & \bar{H}\mathcal{R}(\mathbb{S}^1).
\end{array}$$

Lemma 7.3. *The set of path components $\pi_0(K^1\mathcal{R})$ is an abelian group.*

Proof. The pullback $K^1\mathcal{R}$ is a simplicial permutative category. Therefore $\pi_0(K^1\mathcal{R})$ is an abelian monoid. Switching the two copies of $\bar{H}\mathcal{R}(P\mathbb{S}^1)$ in the defining diagram for $K^1\mathcal{R}$ results in a homotopy inverse which gives $\pi_0(K^1\mathcal{R})$ a group structure. \square

There is a natural pairing

$$K^n\mathcal{R} \times K^m\mathcal{R} \longrightarrow K^{n+m}\mathcal{R}$$

which is induced by

$$\begin{array}{ccc}
\bar{H}\mathcal{R}(X_+^1, \dots, X_+^n) \times \bar{H}\mathcal{R}(X_+^{n+1}, \dots, X_+^{n+m}) & \longrightarrow & \bar{H}\mathcal{R}(X_+^1, \dots, X_+^{n+m}) \\
(C_{\langle S \rangle}, D_{\langle T \rangle}) & \mapsto & (C \otimes D)_{\langle U \rangle}
\end{array}$$

with

$$(C \otimes D)_{(U_1, \dots, U_{n+m})} := C_{(U_1, \dots, U_n)} \otimes D_{(U_{n+1}, \dots, U_{n+m})}.$$

The functors K^n are natural with respect to strictly bipermutative functors between bipermutative categories.

Let I be the category of finite sets and injective functions. Any morphism in I can be expressed as a composition of an order preserving injection with a permutation. For a permutation $\sigma \in \Sigma_n$ we obtain from [EM, §4], that the induced map

$$\sigma: \bar{H}\mathcal{R}(X_+^1, \dots, X_+^n) \longrightarrow \bar{H}\mathcal{R}(X_+^{\sigma^{-1}(1)}, \dots, X_+^{\sigma^{-1}(n)})$$

is an equivalence of categories. Thus it induces an equivalence of n -simplicial categories on $K^n\mathcal{R}$.

Let $i: n \rightarrow n+1$ be the standard inclusion which misses the element $n+1$. Then Elmendorf and Mandell show in their discussion of Extension Functors [EM, §4], that there is an isomorphism of categories

$$i: \bar{H}\mathcal{R}(X_+^1, \dots, X_+^n) \longrightarrow \bar{H}\mathcal{R}(X_+^1, \dots, X_+^n, 1_+)$$

for every n -tuple of pointed sets (X_+^1, \dots, X_+^n) (compare [Sh, p. 380]). This induces a map $K^n\mathcal{R} \rightarrow K^{n+1}\mathcal{R}$ as follows. First of all the maps $\bar{H}\mathcal{R}(X_+^1, \dots, X_+^n) \longrightarrow \bar{H}\mathcal{R}(X_+^1, \dots, X_+^n, 1_+)$ induce a map from $K^n\mathcal{R}$ to the limit of the system $\bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, 1_+)$. The natural maps from 1_+ to $(PS^1)_0 = 1_+$ and $S_0^1 = +$ then yield the desired map to $K^{n+1}\mathcal{R}$.

One can check that these structure maps fit together to give the following result.

Theorem 7.4. *The assignment $n \mapsto K^n\mathcal{R}$ turns $K^\bullet\mathcal{R}$ into an I -graded bimonoidal category.*

Fixing finite pointed sets X_+^1, \dots, X_+^n , $\bar{H}\mathcal{R}(X_+^1, \dots, X_+^n, -)$ is a functor from the category of finite pointed sets to n -fold simplicial categories. Similar to Lemma 7.1 we get that this is a special Γ -space in the sense of Segal (up to some realizations resp. diagonals).

Lemma 7.5. *The canonical inclusion $n \rightarrow n+1$ induces a weak equivalence $K^n\mathcal{R} \rightarrow K^{n+1}\mathcal{R}$ for $n \geq 1$.*

Proof. Note that

$$\tilde{H}\mathcal{R}(1_+) = \lim \bar{H}\mathcal{R}(Y_{i_1}) \sim \Omega \bar{H}\mathcal{R}(S^1)$$

because the natural map is a homology isomorphism of H -spaces (compare [Se, §4]). We know that $K^n\mathcal{R} \cong \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, 1_+)$. As n is at least one, the defining diagram for this limit admits a flip-map and therefore

$$\lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, 1_+) \sim \Omega \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, S^1).$$

An argument similar to the one at the beginning of the proof shows that

$$\Omega \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, S^1) = \lim \Omega \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, S^1) \sim \tilde{H}\mathcal{R}(1_+, \dots, 1_+) = K^{n+1}\mathcal{R}.$$

□

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